IMPACT RESPONSE BASED ON TIMOSHENKO BEAM THEORY

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Abstract

The elastic impact of a translating flexible pole is studied herein. Three scenarios are considered: 1) transverse impact against a rigid stop, 2) longitudinal impact against a flexible column and 3) transverse impact against a flexible column. Based on Timoshenko beam theory, an analytical solution method using mode superposition for the coupled spring-pole or column-pole system is presented. Any physical set of boundary conditions can be accommodated for the pole and the column.

For all cases involving axial impacts, the maximum initial impact force is governed by the local shear deformation in the column and the axial deformation in the pole. However, for transverse impacts, the maximum initial impact force is governed by the local shear deformation in the pole and the column. A simple formula for the maximum initial force is derived and shown to be quite accurate. In no case is the total mass of the pole significant to the initial peak force. Indeed, based on Euler-Bernoulli beam theory the initial impact force is unbounded as the spring stiffness increases whereas Timoshenko beam theory has a clear limiting value for the initial impact force. The impact duration depends on the wave propagation in the pole or the column.

In addition, the energy transfer between kinetic energies and strain energies reveals both the initial dependence on shear deformation and the transfer of the associated energy to bending energy. The energy exchange also shows the importance of the inertia of the column in absorbing a significant part of the initial kinetic energy of the pole. It is shown that the moment of inertia has a negligible effect on the impact force, which is an interesting conclusion because some structural finite element codes use a lumped mass matrix that includes translational masses but not rotational inertias.

For transverse impact, multiple impacts are considered, and the whole collision event is divided into contact phases and separation phases. It is shown that for all cases the maximum contact force occurs during later contact phases and its value can reach up to 1.5 times the peak force in the first contact phase. The impact duration of the first contact phase depends on the shear wave in the pole or the column according to the mass and wave speed ratios. The total impulse on the pole ranges between 1.5-1.8 times the initial momentum of the pole, depending
on the stiffness of the column. The energy exchange during the multiple impacts, while it can be complicated, reveals that for relatively stiff columns the sum of the translational kinetic and bending strain energies of the pole constitutes approximately 90% of the total energy. In all cases considered, relatively little net energy has been transmitted to the column at the time of final separation.

For axial impact, multiple impacts depend on the relative stiffness of the column and the pole, and also on the inertia of the pole. Hence, the entire collision event for the stiffest column is characterized by a single impact. However, for the most flexible column all cases involve multiple impacts. For the case of single impact, most of the kinetic energy of the pole is transferred into axial strain energy in the pole. However, for the multiple impacts, most of the kinetic energy of the pole is transferred into bending energy in the column. The maximum impact force reaches up to 1.9 times the initial peak force and the total impulse reaches up to 1.9 times the initial momentum of the pole. For all cases, the duration of the entire collision event depend mainly on the wave propagation in the pole.

The impact force and duration depend on the type of impact as well as the end boundary conditions of the column. For all cases, the axial impact yields larger impact force than that for the transverse impact according to the stiffness of the column with similar boundary conditions of the column. The stiffer the column, the larger is the impact force and the smaller the impact duration. In addition, free end column yields the smallest impact force and duration. However, pin-end column gives the largest impact duration and fixed-end column gives the largest impact force.

The dynamic amplification factors for shear force and bending moment depend mainly on the stiffness of the column and the inertia of the pole. Cases involving stiffer column and larger pole inertia yield higher dynamic amplification factors. In addition, the dynamic amplification factors significantly increase for cases involving multiple impacts and always reach their maximum values at later impacts.
# Table of Contents

Acknowledgments ........................................................................................................ iii

Abstract ......................................................................................................................... iv

Table of Contents ........................................................................................................ vi

List of Tables ................................................................................................................ x

List of Figures ............................................................................................................... xi

Chapter 1 Introduction ........................................................................................................ 1

1.1 Background ............................................................................................................. 1

1.2 Previous Work ........................................................................................................ 2

1.3 Scope of Work ......................................................................................................... 5

1.4 Outline .................................................................................................................... 5

Chapter 2 Transverse Impact of a Beam on a Flexible Stop ............................................. 8

2.1 Introduction ............................................................................................................. 8

2.2 Physical Systems ................................................................................................... 8

2.3 Timoshenko beam model .................................................................................... 9

2.4 Analytical Solution ............................................................................................... 11

2.4.1 Separation of variables and uncoupling equations of motion .................. 12

2.4.2 Solution of the generalized coordinate function .................................... 13

2.4.3 Solution of the mode shape functions...................................................... 14

2.4.4 Dimensional boundary conditions ............................................................ 16

2.4.5 Dimensional compatibility conditions at the impact point .................... 16

2.4.6 Nondimensional boundary conditions .................................................... 17

2.4.7 Nondimensional compatibility conditions at the impact point ............ 17

2.4.8 Orthogonality condition .......................................................................... 18

2.4.9 Upper bound of the initial impact force ............................................... 20

2.4.10 Impact duration ....................................................................................... 22
2.4.11 Peak Force using Euler-Bernoulli beam theory ........................................ 22
2.4.12 Mechanical Energy .................................................................................... 22

2.5 Results ............................................................................................................. 25
2.5.1 Effect of spring stiffness ............................................................................. 26
2.5.2 Effect of $\tau$ ............................................................................................... 30
2.5.3 Effect of slenderness ratio $\lambda_B$ ............................................................... 31
2.5.4 Effect of $\eta$ ............................................................................................... 33
2.5.5 Asymmetric impact of rotating beam ......................................................... 33
2.5.6 Euler vs. Timoshenko .................................................................................. 35
2.5.7 Energy breakdown and beam vibration ..................................................... 41

2.6 Summary ......................................................................................................... 45

Chapter 3 Beam Response to Longitudinal Impact by a Pole .......................... 47

3.1 Introduction ...................................................................................................... 47
3.2 Physical System ............................................................................................... 48
3.3 Equation of Motion ......................................................................................... 49
3.4 Analytical Solution ......................................................................................... 51
3.4.1 Mode Shapes ............................................................................................... 51
3.4.2 Characteristic Equation ............................................................................. 51
3.4.3 Initial Peak Impact Force .......................................................................... 53
3.4.4 Mechanical Energy .................................................................................... 56

3.5 Results ............................................................................................................. 58
3.5.1 Parameter Range ......................................................................................... 58
3.5.2 Axial Impact of a Pole against a Column ............................................... 59
3.5.3 Comparison of Timoshenko versus Euler Models .................................... 65
3.5.4 Effect of Rotary Inertia ............................................................................. 68
3.5.5 Energy Breakdown ..................................................................................... 69

3.6 Summary ......................................................................................................... 73

Chapter 4 Transverse Impact of a Horizontal Beam on a Vertical Column ....... 75
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>75</td>
</tr>
<tr>
<td>4.2</td>
<td>Physical System</td>
<td>76</td>
</tr>
<tr>
<td>4.3</td>
<td>Equations of Motion</td>
<td>78</td>
</tr>
<tr>
<td>4.4</td>
<td>Analytical Solution</td>
<td>79</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Separation of Variables</td>
<td>79</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Initial Peak Impact Force</td>
<td>82</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Mechanical Energy</td>
<td>83</td>
</tr>
<tr>
<td>4.5</td>
<td>Solution Methodology</td>
<td>85</td>
</tr>
<tr>
<td>4.6</td>
<td>Numerical Results</td>
<td>85</td>
</tr>
<tr>
<td>4.6.1</td>
<td>Physical Dimensions</td>
<td>85</td>
</tr>
<tr>
<td>4.6.2</td>
<td>Force-Time History</td>
<td>87</td>
</tr>
<tr>
<td>4.6.3</td>
<td>Energy-Time History</td>
<td>89</td>
</tr>
<tr>
<td>4.6.4</td>
<td>Impulse-Time History</td>
<td>92</td>
</tr>
<tr>
<td>4.6.5</td>
<td>Energy Decomposition by Modes</td>
<td>94</td>
</tr>
<tr>
<td>4.6.6</td>
<td>3D Energy Density</td>
<td>96</td>
</tr>
<tr>
<td>4.7</td>
<td>Computational Aspects</td>
<td>99</td>
</tr>
<tr>
<td>4.8</td>
<td>Summary</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>Chapter 5 Additional Scenarios</td>
<td>103</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>103</td>
</tr>
<tr>
<td>5.2</td>
<td>Axial Impact</td>
<td>104</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Wood pole hitting concrete column</td>
<td>104</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Wood pole hitting steel column</td>
<td>107</td>
</tr>
<tr>
<td>5.2.3</td>
<td>Wood pole hitting wood column</td>
<td>110</td>
</tr>
<tr>
<td>5.3</td>
<td>Transverse Impact</td>
<td>114</td>
</tr>
<tr>
<td>5.4</td>
<td>Maximum Shear and Bending Envelopes</td>
<td>117</td>
</tr>
<tr>
<td>5.4.1</td>
<td>Envelope versus time</td>
<td>118</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Envelope versus space</td>
<td>123</td>
</tr>
</tbody>
</table>
5.5 Summary ................................................................................................................. 128

Chapter 6 Conclusion .................................................................................................. 131

6.1 Response of a Beam Hitting Transversely a Stop ............................................ 131

6.2 Beam Response to Longitudinal Impact by a Pole ........................................... 132

6.3 Transverse Impact of a Horizontal Beam on a Vertical Column ...................... 133

6.4 Additional Scenarios ............................................................................................ 134
List of Tables

Table 1: Dimensional Properties ................................................................. 60
Table 2: Nondimensional Parameters .......................................................... 60
Table 3: Length Ratios .............................................................................. 60
Table 4: Dimensional Properties ................................................................. 86
Table 5: Nondimensional Parameters .......................................................... 87
Table 6: Maximum dynamic amplification factors for shear force for axial impact ..... 127
Table 7: Maximum dynamic amplification factors for bending moment for axial impact .................................................................................................................. 128
Table 8: Maximum dynamic amplification factors for shear force for transverse impact .................................................................................................................. 128
Table 9: Maximum dynamic amplification factors for bending moment for transverse impact .................................................................................................................. 128
List of Figures

Figure 1: Schematic of beam and stop spring................................................................. 9

Figure 2. Impact zone at time $t^*$ .................................................................................. 21

Figure 3: Impact force for approximately "rigid" impact with $\tau = 10, \eta_2 = 1, \lambda b = 120$ .................................................................................................................. 28

Figure 4: Impact force for a range of $k$, with $\eta_2 = 1, \lambda B = 120$ and $\tau = 10$ ............... 29

Figure 5: Impact force for a range of $k$, with $\eta_2 = 1, \lambda B = 120$ and $\tau = 20/9$ ............ 29

Figure 6: Impact force for a range of $\tau$, with $k = 1200, \eta_2 = 1, \lambda B = 120$ ............... 31

Figure 7: Impact force for a range of $\lambda B$, with $k = 1200, \eta_2 = 1, \tau = 10$ .................... 32

Figure 8: Impact force for a range of $\lambda B$, with $k = 10, \eta_2 = 1, \tau = 10$ ....................... 32

Figure 9: Impact force for $Lr = 120, \tau = 10$ with soft and stiff springs and $\eta_2 = 1$ and $\eta_2 = 0.01$ .................................................................................................................. 33

Figure 10: Impact force for impact at quarter span, $\lambda B = 120, \tau = 10, \eta_2 = 1$ and $k = 1200$ (All plots coincide) .............................................................................................................. 34

Figure 11: Branches of dimensionless group speed $c_g = c_g * /cs$ as a function of the dimensionless wave number $\xi = \xi * r$ for $\tau = 10$ .............................................................. 36

Figure 12: Comparison of impact forces between Timoshenko and Euler-Bernoulli beam theories for different $k$, with $\lambda B = 120, \tau = 10, \eta_2 = 1$ ......................................................... 41

Figure 13: Time evolution of potential and kinetic energy components (log scale) for $\lambda B = 120, \tau = 10, \eta_2 = 1$ and $k = 1200$ ........................................................................................................ 43

Figure 14: Space-time plot of potential energy density for $\lambda B = 120, \tau = 10, \eta_2 = 1$ and $k = 1200$ .................................................................................................................. 44
Figure 15: Space-time plot of kinetic energy density for $\lambda_B = 120$, $\tau = 10$, $\eta^2 = 1$, $k = 1200$
................................................................................................................................................. 45

Figure 16: Schematic of beam/column and pole............................................................................. 49

Figure 17: Impact zone at time $t$ .................................................................................................. 54

Figure 18: Impact force for a wood pole hitting fixed-fixed concrete columns ...................... 62

Figure 19: Impact force for a wood pole hitting fixed-fixed steel columns ......................... 63

Figure 20: Impact force for a wood pole hitting fixed-fixed wood columns......................... 64

Figure 21: Impact force for a wood pole hitting steel columns (left axis) and the end rotation of
the pinned column (right axis) for the case of C2P2 .............................................................. 65

Figure 22: Impact force for a wood pole hitting fixed-end concrete (C), steel (S), and wood (W)
columns for case C1P2 based on Timoshenko (T) and Euler (E) beam theories ..... 67

Figure 23: Impact force for a wood pole hitting a fixed-end wood column for case C1P2 for
different ratios of the contact stiffness to the axial pole stiffness, for Timoshenko (T) and
Euler (E) theories ....................................................................................................................... 67

Figure 24: Effect of rotary inertia on impact force for $L/r=120$ and $\tau = 20/9$ with $\eta^2 = 1$ and
$\eta^2 = 0.01$ .................................................................................................................................. 69

Figure 25: Shear and bending strain energy in the wood column for case C1P2 ............ 71

Figure 26: Component energies and total energy for the case C1P2 with a wood column72

Figure 27: Time variation of energy densities for case C1P2 with a wood column........ 73

Figure 28: Schematic of pole and column .................................................................................... 78

Figure 29: Impact force time histories ......................................................................................... 89

Figure 30: Component energies and total energy ........................................................................ 92
Figure 31: Impulse time histories a wood pole hitting columns of different lengths and materials ................................................................. 94

Figure 32: Modal contribution of energy of the pole for short concrete column (left) and long wood column (right) ........................................................................................................ 96

Figure 33: Time variation of energy densities of the pole for short concrete column (left) and long wood column (right) ........................................................................................................ 99

Figure 34: Force time history for 3.6 m concrete column obtained from analytical and FEA solution ........................................................................................................ 101

Figure 35: Component energies and total energy for a wood pole hitting axially a concrete column ........................................................................................................ 105

Figure 36: Impulse for a wood pole hitting axially a concrete column ..................... 106

Figure 37: Force-time history for case C2P2 of a wood pole hitting axially a steel column 107

Figure 38: Energy components and total energy for a wood pole hitting axially a steel column ........................................................................................................ 109

Figure 39: Impulse for a wood pole hitting axially a steel column ......................... 110

Figure 40: Impact force for a wood pole hitting axially a wood column...................... 111

Figure 41: Energy components and total energy for a wood pole hitting axially a wood column ........................................................................................................ 113

Figure 42: Impulse for a wood pole hitting axially a wood column............................ 114

Figure 43: Impact force for 9 m wood pole hitting transversely a 6 m wood column... 115

Figure 44: Energy components and total energy for a 9 m wood pole hitting transversely a 6 m wood column ........................................................................................................ 116

Figure 45: Impulse for a 9 m wood pole hitting transversely a 6 m wood column ....... 117
Figure 46: Maximum shear force (left) and bending moment (right) amplification factors for axial impact................................................................. 119

Figure 47: Maximum shear force (left) and bending moment (right) amplification factors for transverse impact ................................................................. 122

Figure 48: Maximum shear force (left) and bending moment (right) amplification factors for axial impact................................................................. 125

Figure 49: Maximum amplification factors for transverse impact ......................... 127
Chapter 1

Introduction

1.1 Background

A significant threat to structures in the tsunami inundation zone is impact from debris driven by the tsunami flow (NRC 2004). A proper characterization of these forces is especially important to life-safety related to vertical tsunami evacuation shelters (FEMA 2012). Debris driven by tsunami waves can cause catastrophic damages to coastal buildings. Debris is composed of different materials, such as building fragments, boats, vehicles, wood poles, docks, and shipping containers. In March 2011, huge tsunami waves hit the coast of Japan resulting in approximately 16,000 deaths, 6,000 injured, and damage to the buildings. 5 million tons of debris swept in the Pacific Ocean, 70% of them sank near the coast of Japan. The remaining 30% of the debris moved in the ocean and some of them appeared in the coasts of the United States and Canada in late 2011 (Alaska 2013; Toro 2012).

On September 29, 2009, an earthquake of magnitude 8.0 hit the islands of Samoa. Consequently, destructive tsunami waves were generated and washed the islands, causing damages to the buildings and fatalities. It has been reported that almost 137 people died and 310 were injured. In addition, some of the buildings suffered damages due to hydrodynamic and debris impact loads (Robertson et al. 2010).

On December 26, 2004 a huge earthquake hit Sumatra and was followed by devastating tsunami waves in the Indian Ocean. Final estimation showed that 226,226 people died, including 49,648 missing people, and nearly two million people were forced to evacuate. One of the main causes of damage to buildings was debris impact, especially exposed structures (Rossetto et al. 2006).

Hurricane Katrina is considered one of the deadliest storms that hit the United States. Approximately 1,836 people died and millions became homeless (Zimmermann 2012). In August 2005, the storm reached its peak (storm category 5) before it hit the Gulf Coast of Louisiana, Mississippi, and Alabama. It lost some of its strength at the time of landfall in
Louisiana/ Mississippi border; however, the storm surge did not abate rapidly. Many structures experienced significant damages due to floating or mobile debris, such as shipping containers, boats, and barges (Robertson et al. 2007).

From the examples mentioned above, it is clear that debris impact can cause profound damage to coastal structures and loss of lives. The situation would be even more dramatic if dangerous chemical or radioactive substances were to spread in the surroundings. Unfortunately, many coastal areas are not designed to resist the damages caused by tsunamis. Relatively little research has been devoted to tsunami-driven debris, although recent tsunamis have illustrated the potential for structural damage from such debris. Low velocity impact of high mass, water-driven debris on civil-type structures has received attention primarily related to flood-borne woody debris (Haehnel and Daly 2002; Matsutomi 2009), barge collision on bridge piers (Consolazio et al. 2006; Consolazio and Cowan 2005; Consolazio et al. 2009) and navigation locks (Arroyo-Caraballo and Ebeling 2006).

The objective of this work is to improve our understanding of, and predictive capabilities for, tsunami-driven debris impact forces on structures. The focus is on low velocity and high mass woody debris, which is one of the more prevalent substantial debris in tsunami flow for industrial and densely populated coastal areas. Impact forces specified by current codes and standards are based on rigid body dynamics, while the work herein includes the flexibility of the debris as well as the structural member (column) for more accurate prediction of impact forces.

1.2 Previous Work

Many studies on the impact response of beams have been carried out. The studies are often motivated by a number of important scenarios associated with mechanical equipment, such as piping systems, heat exchangers and valves, and fall into three primary categories: a (typically rigid) mass impacting a beam, a beam hitting a rigid or flexible stop, and two beams hitting each other. An early study was Timoschenko (1914), who used an Euler beam, a Hertzian contact model, and mode superposition to obtain a solution for a beam hit by a mass. Boley and Chao (1955) used the Laplace transform to study the behavior of a semi-infinite beam under transverse impact. The impact was modeled as a “sudden load” and numerical integration was used to compute the definite integrals in the analytical solution. These authors briefly compared
numerical results for Timoshenko and Euler-Bernoulli beam theories under these conditions, and obtained poor correlation for the initial force propagation. Goyder and Teh (1989) used a single-degree-of-freedom spring model to evaluate repeated impacts of a tube in a loose support. In an attempt to reduce the computational complexity of impact simulation, Evans et al. (1991) developed an efficient numerical integration scheme for a mass hitting a round Kirchhoff plate, based on a Hertzian contact model and mode superposition.

Xing et al. (2002) and Vinayaravi et al. (2013) studied impact of a rigid mass on a beam. Vinayaravi et al. (2013) investigated the damping as a result of repeated impact of a mass on a cantilever beam at the tip, including repeated impacts. The physical system was modeled as a two degree-of-freedom spring-mass-damper system. They observed that the ‘damping’ depends on the number of effective impacts (impact with higher relative velocity between the beam and the mass) and not on the total number of impacts. Xing et al. (2002) used Timoshenko beam theory and modal superposition to study the impact of a rigid mass hitting a beam at midspan. They used conservation of linear momentum to estimate the initial impact force and observed that the initial value of the impact force depends on the shear wave propagation in the beam and not on the flexural wave propagation. However, when the flexural wave returns to the impact point, the impact force starts to have small vibrations, and when the shear wave returns to the impact point, separation occurs between the mass and the beam; hence, there was only one impact. Ervin and Wickert (2007) used modal superposition to model multiple impacts of a mass on a fixed Euler beam hitting a compliant stop.

Yin et al. (2007) and Wang and Kim (1996) studied the transient behavior of a cantilever beam hitting a rod. Both papers used Euler-Bernoulli beam theory and modal superposition to solve the beam equation of motion and the one-dimensional wave equation for the axial response of the rod. In Yin et al. (2007), the beam tip was subjected to a periodic force. The repeated ‘impact’ is divided into three phases: pre-impact, impact, and separation. The results showed that the impacts induce high frequency response. Wang and Kim (1996) observed that representing the valve stop as a spring yields good results for a short stop; however, for a relatively long stop the results deviate greatly from the exact solution in which the inertia of the rod is considered. Wagg and Bishop (2002) considered a cantilever Euler beam hitting a rigid stop. They assumed a
coefficient of restitution and focused on pre- and post-impact, rather than on the response during impact. Again, normal modes were used.

Packzkowski (2012) studied the axial impact of a projectile hitting a rigid wall. The author was able to predict the impact force and duration using one-dimensional wave equation to model the projectile. The results showed that the impact force does not depend on the total mass of the projectile and that the impact duration is the time taken by the axial wave to propagate in the projectile and come back to the impact point. Khowitar et al. (2014) studied longitudinal (axial) impact of a wood pole against a column. The one dimensional axial wave equation of motion was used for the pole and Timoshenko beam theory was used for the column. A stiff spring was placed between the pole and the column to model a finite contact stiffness. The modal superposition method was used to solve for the analytical response. A closed form equation for the instantaneous peak impact force was also derived. It was shown that the instantaneous impact force is dominated by the shear wave in the column, and hence, Euler-Bernoulli beam theory yields inaccurate results except for quite low values of contact stiffness. A study on the energy transfer during the impact was also conducted.

Dorogy and Rittel (2008) conducted experiments and a three-dimensional finite element analysis of a free-free beam subjected to transverse impact by a Hopkinson bar at four locations along the beam. They considered elastic-plastic behavior of the beam, with bilinear isotropic hardening as well as geometrical nonlinearity, in the numerical analysis. They observed that most of the plastic deformation occurs very soon after impact. Symmetrical impact has the maximum impact force and dissipated energy due to plastic deformation. Approximately 76% of the initial energy is converted into plastic deformation and approximately 21% is transferred into kinetic energy of the beam. Langley (2012) developed a statistical solution for randomly impacting objects assuming that the force time history is approximated by a quarter sine curve. He was able to predict the impact force and duration of two identical Euler beams hitting each other with a non-linear spring at the impact point. Ervin (2009) studied multiple impacts of two beams subject to base excitation at the supports. Euler-Bernoulli beam theory and modal superposition were used to obtain the response. A spring was placed between the beams at the impact point to account for the contact stiffness. She studied the effect of different parameters on the multiple impact response spectra. Christoforou and Yigit (1998) found the main parameters
on which the impact force depends for two special impact cases. The first case was half space impact, in which the target mass is very large compared to the mass of the impactor. In this case the flexural wave is not reflected back to the impact point from the boundary of the target and the deformation is localized around the impact point. Therefore, they used an infinite beam and Timoshenko beam theory to find the peak force. They noticed that the main parameter affecting the impact force is the normalized impact velocity. In the second case, the mass of the impactor was very large compared to the target. In this case the problem was quasi-static, in which a load is applied at the impact point. The mass of the target is neglected and the problem can be seen as a single degree of freedom system with contact stiffness and static beam stiffness. They found that the only parameter affecting the impact force in this case is the ratio of the beam static stiffness to the contact stiffness.

1.3 Scope of Work

The objective of this work is to improve our understanding of, and predictive capabilities for, tsunami-driven debris impact forces on structures. A low velocity, high mass, in-air debris impact theory based on flexible body dynamics is developed.

Significant amount of work has been carried out on the beam impact. However, most of this work is based on the Euler-Bernoulli theory to obtain the response, which is proved later to be inadequate and yields inaccurate results for the axial and the transverse impact. It will be shown in this study that the impact force depends mainly on the shear deformation of the beam for transverse impact, and hence Timoshenko beam theory is used to account for the shear deformation and the rotary inertia. On the other hand, the multiple impacts for beam-beam transverse impact have not received much attention, but will be discussed in detail herein.

1.4 Outline

Three different scenarios are considered in the following chapters: 1) a pole hitting transversely a flexible stop, 2) a pole hitting longitudinally a column, and 3) a pole hitting transversely a column. Each of the chapters are written to stand essentially by themselves. Therefore, they are a complete statement of the problem, and therefore there is some overlap between the chapters.
In Chapter 2, the transverse impact of a pole hitting a massless spring is discussed. In this case, the pole can have arbitrary initial translational and rotational velocities, and both Timoshenko and Euler-Bernoulli beam theories are considered. A detailed parameter study is carried out to investigate the effect of the nondimensional parameters on the impact force time history.

In Chapter 3, axial impact of a pole hitting a flexible column is investigated. The column can have any arbitrary end boundary conditions. A simple design formula for the peak value of the impact force and the impact duration is derived. Moreover, the energy exchange is discussed as it gives insight on the behavior of the impact force.

In Chapter 4, transverse impact of a pole hitting a column is investigated. The behavior of the impact force and relevant phenomena are studied for multiple impacts. Simple formulas for the peak impact force and duration are derived. To understand better the behavior of the impact force during the multiple impacts, the energy exchange among various components as well as the force impulse are investigated.

In Chapter 5, additional scenarios for axial and transverse impacts that supplement the results in the previous two chapters are considered. The axial impact simulations presented in Chapter 3 only considered the first contact phase. In this chapter, the same scenarios are simulated but through the entire impact event, capturing the multiple impacts, if any. The maximum impact force, which normally does not occur during the first contact phases, is sought. Moreover, the main differences between cases that exhibit single and multiple impacts will be clarified from the energy exchange. For transverse impact, free-end and pin-end column boundary conditions are considered to supplement the fixed-end boundary conditions in Chapter 4. The free-end condition, while not particularly relevant for a pole hitting a column, has some interest in the transverse impact of two poles, for example.

In an attempt to assess the values of the initial peak force obtained herein for design purposes, it is useful to give an insight on whether applying this force as a static load can bound the dynamic response. The values of the maximum shear force and bending moment are obtained by applying a static load of a value equal to the initial peak force at mid-span of a fixed-end column.
These values are then compared to the values of the shear force and the bending moments from the dynamic analyses.

Finally, in Chapter 6, conclusions of the main concepts and major findings are presented.

The results of this study will contribute to improving community resilience to tsunamis. The results have the potential to impact significantly the specification of design forces for debris in codes and standards. Debris impact is an especially important design consideration for tsunami shelters, fuel and chemical storage tanks, and port and industrial facilities, all of which may unavoidably be located in tsunami inundation zones. In addition, the results will be applicable to hurricane-driven, water-borne debris, and to some extent to barge and ship collisions of bridge piers, docks, and navigation locks. The models are a significant improvement over that which is currently used for design forces.
Chapter 2

Transverse Impact of a Beam on a Flexible Stop

2.1 Introduction

The investigation herein is motivated by the threat of debris strikes on buildings, such as by a floating log propelled by a tsunami or hurricane storm surge and striking a column. As such, the focus is on the impact force during single impact. A previous study (Kobayashi et al. 2012; Paczkowski et al. 2012) considered axial impact, and this study focuses on transverse impact. In this chapter, an initial model is investigated that involves a free-free beam with initial translational and rotational velocities striking a flexible spring anywhere within the span. The response of the beam is investigated to guide the models adopted in subsequent chapters for more accurate models of the pole-column impact. The discussion is restricted to linear impact response. An approach similar to Xing et al. (2002) is followed to develop analytical solutions for both Timoshenko and Euler beam theories, but the approach adopted herein is more general and covers a broader range of problems. The range of applicability of the two beam theories is investigated. Of primary interest is the high resolution capture of the impact force time history.

This chapter is organized as follows. First, the governing equations are defined and nondimensionalized. Then the analytical solutions, which are based on superposition of the normal modes of vibration, are developed. A parameter study is conducted to explore the effect of several key nondimensional parameters on the impact behavior, and to evaluate the range of validity of the Euler beam model for this application. The final section contains conclusions from the present study.

2.2 Physical Systems

A schematic of the system is shown in Figure 1. The figure indicates a pole with an initial translational and angular velocity striking a spring with a stiffness \(k^*\). The pole is assumed to be elastic and homogenous, with constant cross-sectional properties.
Figure 1: Schematic of beam and stop spring

2.3 Timoshenko beam model

Under the assumptions above, the equations of motion for a Timoshenko beam model can be written as

\[ \rho A \frac{\partial^2 w^*}{\partial t^2} - \kappa GA \left( \frac{\partial^2 w^*}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) + k_s^* \delta^* = 0 \]  \hspace{1cm} (2.1)

\[ I_m \frac{\partial^2 \theta}{\partial t^2} - \kappa GA \left( \frac{\partial w^*}{\partial x} - \theta \right) - EI \frac{\partial^2 \theta}{\partial x^2} = 0 \]  \hspace{1cm} (2.2)

where \( w^*(x^*, t^*) \) is the transverse displacement of the pole, \( \theta(x^*, t^*) \) is the rotation of the cross section of the pole, \( \rho \) is the mass density, \( A \) is the cross-sectional area, \( E \) is the Young's elastic modulus, \( G \) is the shear modulus, \( I \) is the cross-sectional area moment of inertia, \( I_m = \eta^2 \rho I \) is the sectional rotary inertia of the beam, \( \kappa \) is the shear coefficient, \( k_s^* \) is the spring stiffness, \( \delta^* \) is the Kronecker delta, and \( L^* = L_1^* + L_2^* \) is the length of the pole. The impact point can be at any arbitrary point along the pole (\( x^* = 0 \)).

The boundary conditions at the beam ends are null shear force and null moment. The shear force at the free ends is specified by

\[ \kappa GA \left( \frac{\partial w^*}{\partial x} - \theta \right) \bigg|_{x^* = L_1^*} = 0 \]

\[ \kappa GA \left( \frac{\partial w^*}{\partial x} - \theta \right) \bigg|_{x^* = L_2^*} = 0 \]  \hspace{1cm} (2.3)

and the bending moment at the free ends is specified by
\[ EI \left( \frac{\partial \theta}{\partial x^*} \right)_{x^* = -L_i} = 0 \]
\[ EI \left( \frac{\partial \theta}{\partial x^*} \right)_{x^* = -L_i} = 0 \]  

(2.4)

At impact, the beam’s center of mass is assumed to move at a speed of \( v_0^* \), while the beam rotates at a rate of \( \omega_0^* \).

To reduce the number of parameters while providing broader scope for each solution, these model equations are cast in nondimensional form using the same nondimensionalization as used by Xing et al. (2002):

\[ x = x^* / r \]
\[ w = w^* / r \]
\[ L = L^* / r = \lambda_B \]
\[ L_{1,2} = L_{1,2}^* / r \]
\[ t = t^* c_0 / r \]
\[ k_x = k_x^* / (EA / r) \]

(2.5)

where \( r = \sqrt{I/A} \) is the radius of gyration of the beam section and \( c_0 = \sqrt{E/\rho} \) is the speed of sound in the beam.

The nondimensionalization in Eqs. (2.5), based on \( r \) and \( r / c_0 \), is convenient to develop the solution. However, the length scale \( L^* \) and time scale \( L^* / c_s \), where \( c_s = \sqrt{KS/\rho} \) is the shear speed in the beam, are more meaningful physically. Therefore, we will use the more physically relevant parameters to discuss the results.

The space derivatives of the kinematic variables are
The time derivatives of the kinematic variables are

\[
\frac{\partial w}{\partial t} = \frac{\partial w_r}{\partial t} = \frac{\partial w_r}{\partial t} = \frac{\partial w}{\partial t} r = \frac{\partial w}{\partial t} r
\]

\[
\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial t^2} c_0 = \frac{\partial^2 w}{\partial t^2} c_0 = \frac{\partial^2 w}{\partial t^2} c_0 = \frac{\partial^2 w}{\partial t^2} c_0
\]

\[
\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t} c_0 = \frac{\partial \theta}{\partial t} c_0 = \frac{\partial \theta}{\partial t} c_0
\]

\[
\frac{\partial^2 \theta}{\partial t^2} = \frac{\partial^2 \theta}{\partial t^2} c_0 = \frac{\partial^2 \theta}{\partial t^2} c_0 = \frac{\partial^2 \theta}{\partial t^2} c_0
\]

These parameters can be used to express the equations of motion in nondimensional form. Substituting the non-dimensional parameters in Eq. (2.1) results in

\[
\frac{\partial^2 w}{\partial t^2} - \frac{1}{\tau} \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) = 0
\]

(2.8)

where \( \tau = E \left( \kappa G \right) = c_0^2 / c_s^2 \). Substituting the non-dimensional parameters in Eq. (2.2) results in

\[
\eta \frac{\partial^2 \theta}{\partial t^2} - \frac{1}{\tau} \left( \frac{\partial w \theta}{\partial x} \right) - \frac{\partial^2 \theta}{\partial x^2} = 0
\]

(2.9)

### 2.4 Analytical Solution

An analytical solution to the Timoshenko model above is obtained using the method of modal superposition.
2.4.1 Separation of variables and uncoupling equations of motion

Because of the discontinuity in the shear force at the impact point, the displacement and rotation fields are divided into two parts:

\[ w = \begin{cases} 
  w^-, & -L_1 \leq x \leq 0 \\
  w^+, & 0 \leq x \leq L_2 
\end{cases} \]

\[ \theta = \begin{cases} 
  \theta^-, & -L_1 \leq x \leq 0 \\
  \theta^+, & 0 \leq x \leq L_2 
\end{cases} \]

Each of the displacements and the rotations are functions of \( x \) and \( t \). These displacements can be expressed as the product of two functions; one is dependent on \( x \) and the other is dependent on \( t \):

\[ w(x,t) = \phi(x)q(t) \]
\[ \theta(x,t) = \psi(x)q(t) \] (2.10)

where \( \phi(x) \) and \( \psi(x) \) are the mode shape functions and \( q(t) \) is the generalized coordinate function.

Substitution of Eqs. (2.10) into Eq. (2.8) yields

\[ \phi \ddot{q} - \frac{1}{r} (\phi^* q - \psi^* q) = 0 \]
\[ r \phi \ddot{q} = q (\phi^* - \psi^*) \]
\[ \frac{\ddot{q}}{q} = \frac{(\phi^* - \psi^*)}{r \phi} = -\omega^2 \]

where \( \omega \) is the non-dimensional natural frequency of the beam and the dimensional natural frequency \( \omega^* = \omega \times c_0 / r \). Thus,

\[ \ddot{q} + \omega^2 q = 0 \] (2.11)
\[ \phi^* - \psi' + \omega^2 \tau \phi = 0 \] (2.12)

Substitution of Eqs. (2.10) into Eq. (2.9) results in
\[
\psi'' + \frac{1}{\tau} \phi' + \left( \omega^2 \eta^2 - \frac{1}{\tau} \right) \psi = 0
\]  
(2.13)

Taking the first derivative of Eq. (2.12) yields
\[
\phi'' - \psi'' + \omega^2 \tau \phi' = 0
\]  
(2.14)

Rearranging terms in Eq. (2.13) and taking the second derivative with respect to \( x \) results in
\[
\phi' = -\tau \psi'' - \left( \omega^2 \eta^2 \tau - 1 \right) \psi
\]
\[
\phi''' = -\tau \psi^{(4)} - \left( \omega^2 \eta^2 \tau - 1 \right) \psi''
\]  
(2.15)

Substitution of Eqs. (2.15) into Eq. (2.14) yields the differential equation for the rotational mode shape function
\[
\psi^{(4)} + \left( \omega^2 \eta^2 + \omega^2 \tau \right) \psi'' + \left( \omega^4 \eta^2 \tau - \omega^2 \right) \psi = 0
\]  
(2.16)

Following the same procedure, the differential equation for the translational mode shape function is
\[
\phi^{(4)} + \left( \omega^2 \eta^2 + \omega^2 \tau \right) \phi'' + \left( \omega^4 \eta^2 \tau - \omega^2 \right) \phi = 0
\]  
(2.17)

### 2.4.2 Solution of the generalized coordinate function

Let \( q = e^{i \omega t} \). Substitution into Eq. (2.11) results in
\[
v^2 + w^2 = 0
\]

The roots of the above equation are \( \zeta_{1,2} = \pm i \omega \), and thus the solution becomes
\[
q = c_1 e^{i \omega t} + c_2 e^{-i \omega t}
\]

Using the Euler formulas, the solution can be written as
\[
q = c_1 \left( \cos \omega t + i \sin \omega t \right) + c_2 \left( \cos \omega t - i \sin \omega t \right)
\]
\[
= (c_1 + c_2) \cos \omega t + (i c_1 - i c_2) \sin \omega t
\]

Thus,
\[ q = C_1 \cos \omega t + C_2 \sin \omega t \]  \hfill (2.18)

2.4.3 Solution of the mode shape functions

Let \( \phi = e^{\lambda x} \). Substitution into Eq. (2.17) results in

\[ \lambda^4 + \left[ \omega^2\eta^2 + \omega^2\tau \right] \lambda^2 + \omega^4\eta^2\tau - \omega^2 = 0 \]

The roots of this equation are

\[ \lambda_{1,2} = \pm i \sqrt{\omega \sqrt{\omega^2 \left( \eta^2 - \tau \right)^2 + 4 + \omega^2 \left( \eta^2 + \tau \right)}} / \sqrt{2} \]
\[ = \pm i \beta \]

\[ \lambda_{3,4} = \pm \sqrt{\omega \sqrt{\omega^2 \left( \eta^2 - \tau \right)^2 + 4 - \omega^2 \left( \eta^2 + \tau \right)}} / \sqrt{2} \]
\[ = \pm \alpha \]

The mode shape function then becomes

\[ \phi(x) = a_1 e^{\beta x} + a_2 e^{-\beta x} + A_1 e^{\alpha x} + A_2 e^{-\alpha x} \]

Using the Euler formulas in the mode shapes function gives

\[ \phi(x) = a_1 \left( \cos \beta x + i \sin \beta x \right) + a_2 \left( \cos \beta x - i \sin \beta x \right) + A_1 e^{\alpha x} + A_2 e^{-\alpha x} \]
\[ = (a_1 + a_2) \cos \beta x + (a_1i - a_2i) \sin \beta x + A_1 e^{\alpha x} + A_2 e^{-\alpha x} \]

Thus, the mode shape function for the displacement is

\[ \phi(x) = A_1 \cos \beta x + A_2 \sin \beta x + A_3 e^{\alpha x} + A_4 e^{-\alpha x} \] \hfill (2.19)

Using the same procedure, the mode shape function for the rotation becomes

\[ \psi(x) = B_1 \cos \beta x + B_2 \sin \beta x + B_3 e^{\alpha x} + B_4 e^{-\alpha x} \] \hfill (2.20)

A relationship can be found between \( B_i \) and \( A_i \) by substituting the mode shape functions in Eq. (2.12):
\[
\left(-\beta^2 A_1 \cos \beta x - \beta^2 A_2 \sin \beta x + \alpha^2 A_3 e^{\alpha x} + \alpha^2 A_4 e^{-\alpha x}\right) - \\
\left(-\beta B_1 \sin \beta x + \beta B_2 \cos \beta x + \alpha B_3 e^{\alpha x} - \alpha B_4 e^{-\alpha x}\right) + \\
\omega^2 \tau (A_1 \cos \beta x + A_2 \sin \beta x + A_3 e^{\alpha x} + A_4 e^{-\alpha x}) = 0
\]

Rearranging terms

\[
\left(-\beta^2 A_1 + \beta B_2 + \omega^2 \tau A_1\right) \cos \beta x + \left(-\beta^2 A_2 - \beta B_1 + \omega^2 \tau A_2\right) \sin \beta x + \\
\left(\alpha^2 A_3 + \alpha B_3 + \omega^2 \tau A_3\right) e^{\alpha x} + \left(\alpha^2 A_4 + \alpha B_4 + \omega^2 \tau A_4\right) e^{-\alpha x} = 0
\]

Eq. (2.21) is satisfied only if each coefficient of \(\cos \beta x\), \(\sin \beta x\), \(e^{\alpha x}\), and \(e^{-\alpha x}\) is null. Hence,

\[
\begin{align*}
-\beta^2 A_1 + \beta B_2 + \omega^2 \tau A_1 &= 0 \\
-\beta^2 A_2 - \beta B_1 + \omega^2 \tau A_2 &= 0 \\
\alpha^2 A_3 + \alpha B_3 + \omega^2 \tau A_3 &= 0 \\
\alpha^2 A_4 + \alpha B_4 + \omega^2 \tau A_4 &= 0
\end{align*}
\]

This leads to

\[
\begin{align*}
B_2 &= \frac{\beta^2 - \omega^2 \tau}{\beta} A_1 = \kappa_\beta A_1 \\
B_1 &= \frac{-\beta^2 + \omega^2 \tau}{\beta} A_2 = -\kappa_\beta A_2 \\
B_3 &= \frac{-\alpha^2 - \omega^2 \tau}{\alpha} A_3 = -\kappa_\alpha A_3 \\
B_4 &= \frac{\alpha^2 + \omega^2 \tau}{\alpha} A_4 = \kappa_\alpha A_4
\end{align*}
\]

where

\[
\kappa_\beta = \frac{\beta^2 - \omega^2 \tau}{\beta}, \quad \kappa_\alpha = \frac{\alpha^2 + \omega^2 \tau}{\alpha}
\]

Hence, the mode shapes on each side of the discontinuity at \(x = 0\) are

\[
\begin{align*}
\{\phi^\pm\} &= \begin{cases}
A_1^\pm \cos \beta x + A_2^\pm \sin \beta x + A_3^\pm e^{\alpha x} + A_4^\pm e^{-\alpha x}
\end{cases} \\
\{\psi^\pm\} &= \begin{cases}
\kappa_\beta (A_1^\pm \sin \beta x - A_2^\pm \cos \beta x) + \kappa_\alpha (A_3^\pm e^{\alpha x} - A_4^\pm e^{-\alpha x})
\end{cases}
\end{align*}
\]

(2.22)
Note that $\beta$ is always real, but $\alpha$ can be real or imaginary. A special mode shape must be considered when $\omega \rightarrow 1/(\eta \sqrt{\tau})$, $\alpha \rightarrow 0$, in which the mode shapes can be obtained by taking the limit of Eq. (2.22). We note that these eigenfunctions are consistent with Xing et al. (2002), who considered only the case $\eta = 1$.

2.4.4 Dimensional boundary conditions

The dimensional boundary conditions for the shear force at the ends of the beam are

$$\kappa GA \left( \frac{\partial \phi^x}{\partial x} - \psi^x \right) \bigg|_{x = L_i} = 0$$

$$\kappa GA \left( \frac{\partial \phi^x}{\partial x} - \psi^x \right) \bigg|_{x = L_2} = 0$$

The dimensional boundary conditions for the bending moment at the ends of the beam are

$$EI \left( \frac{\partial \psi^x}{\partial x} \right) \bigg|_{x = L_i} = 0$$

$$EI \left( \frac{\partial \psi^x}{\partial x} \right) \bigg|_{x = L_2} = 0$$

2.4.5 Dimensional compatibility conditions at the impact point

The dimensional compatibility condition for the displacement at the impact point is

$$\phi^- (0) = \phi^+ (0)$$

The dimensional compatibility condition for the rotation at the impact point is

$$\psi^- (0) = \psi^+ (0)$$

The dimensional compatibility condition for the bending moment at the impact point is

$$EI \left( \frac{\partial \psi^x}{\partial x} \right) \bigg|_{x^* = 0} = EI \left( \frac{\partial \psi^x}{\partial x} \right) \bigg|_{x^* = 0}$$

The difference in the shear forces at the impact point equals the force in the spring, thus the
dimensional compatibility condition for the shear force at the impact point is

\[
\kappa G A \left( \frac{\partial \phi^+}{\partial x} - \psi^+ \right)_{x^* = 0} - \kappa G A \left( \frac{\partial \phi^-}{\partial x} - \psi^- \right)_{x^* = 0} = k^* \phi^*_w (0)
\]

in which \( \phi^*_w (0) = (\phi^+ (0) + \phi^- (0)) / 2 \).

2.4.6 Non-dimensional boundary conditions

The nondimensional boundary conditions for the shear force at the ends of the beam are

\[
\left. \left( \frac{\partial \phi^-}{\partial x} - \psi^- \right) \right|_{x = -L_1} = 0
\]

\[
\left. \left( \frac{\partial \phi^+}{\partial x} - \psi^+ \right) \right|_{x = L_2} = 0
\]

The nondimensional boundary conditions for the bending moments at the ends of the beam are

\[
\left. \frac{\partial \psi^-}{\partial x} \right|_{x = -L_1} = 0
\]

\[
\left. \frac{\partial \psi^+}{\partial x} \right|_{x = L_2} = 0
\]

2.4.7 Non-dimensional compatibility conditions at the impact point

The nondimensional compatibility condition for the displacement at the impact point is

\[
\phi^- (0) = \phi^+ (0)
\]

The nondimensional compatibility condition for the rotation at the impact point is

\[
\psi^- (0) = \psi^+ (0)
\]

The nondimensional compatibility condition for the bending moment at the impact point is

\[
\left. \frac{\partial \psi^-}{\partial x} \right|_{x = 0} = \left. \frac{\partial \psi^+}{\partial x} \right|_{x = 0}
\]

The nondimensional compatibility conditions for the shear force at the impact point is
2.4.8 Orthogonality condition

The dimensional orthogonality condition can be written as

\[
\left( \frac{\partial \phi^+}{\partial x} - \frac{\partial \phi^-}{\partial x} \right)\bigg|_{x=0} = \frac{k_s}{\kappa GA} \phi_{av}^*(0)
\]

\[
= \frac{k_s (EA/r)}{\kappa GA} \phi_{av}^*(0) r
\]

\[
= \tau k_s \phi_{av}^*(0)
\]

\[
\int_{-L}^{0} (m \phi^+ \phi^- + I_m \psi^- \psi^-) dx + \int_{0}^{L} (m \phi^- \phi^+ + I_m \psi^+ \psi^+) dx = 0
\]

where \( m = \rho A \) is the mass per unit length of the beam. The nondimensional form of the orthogonality condition would be

\[
\int_{-L}^{0} (\phi^+ \phi^- + \eta^2 \psi^- \psi^-) dx + \int_{0}^{L} (\phi^- \phi^+ + \eta^2 \psi^+ \psi^+) dx = 0
\]

The coefficients of the mode shape functions \( A_i^+ \) and \( A_i^- \), \( i = 1, 2, 3, 4 \) are 8 coefficients, which is the same as the number of boundary and compatibility conditions. Hence, the coefficients can be determined exactly by enforcing satisfaction of the boundary and compatibility conditions.

The four boundary conditions and four compatibility equations result in a system of equations in the eight coefficients plus the natural frequencies \( \omega \). For there to be a non-null solution, the characteristic equation must be zero:
\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\
  a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & a_{35} & a_{36} & a_{37} & a_{38} \\
  0 & 0 & 0 & 0 & a_{45} & a_{46} & a_{47} & a_{48} \\
  0 & \kappa_\beta & -\kappa_\alpha & \kappa_\alpha & 0 & -\kappa_\beta & \kappa_\alpha & -\kappa_\alpha \\
  -1 & 0 & -1 & -1 & 1 & 0 & 1 & 1 \\
  \beta\kappa_\beta & 0 & \alpha\kappa_\alpha & \alpha\kappa_\alpha & -\beta\kappa_\beta & 0 & -\alpha\kappa_\alpha & -\alpha\kappa_\alpha \\
  -\frac{1}{2}(k\tau) & -\beta & -\alpha - \frac{k\tau}{2} & \kappa_\alpha - \frac{k\tau}{2} & -\frac{1}{2}(k\tau) & \beta & \alpha - \frac{k\tau}{2} & -\alpha - \frac{k\tau}{2}
\end{pmatrix}
\]

with

\[a_{11} = (\beta + \kappa_\beta) \sin(L_1\beta); \quad a_{12} = (\beta + \kappa_\beta) \cos(L_1\beta); \quad a_{13} = e^{-L_1\alpha}(\alpha - \kappa_\alpha); \quad a_{14} = e^{L_1\alpha}(\kappa_\alpha - \alpha);
\]
\[a_{21} = \beta\kappa_\beta \cos(L_1\beta); \quad a_{22} = -\beta\kappa_\beta \sin(L_1\beta); \quad a_{23} = e^{-L_1\alpha}\alpha\kappa_\alpha; \quad a_{24} = e^{L_1\alpha}\alpha\kappa_\alpha; \quad a_{35} = -(\beta + \kappa_\beta) \sin(L_2\beta);
\]
\[a_{36} = (\beta + \kappa_\beta) \cos(L_2\beta); \quad a_{37} = e^{-L_1\alpha}(\alpha - \kappa_\alpha); \quad a_{38} = e^{-L_1\alpha}(\kappa_\alpha - \alpha); \quad a_{45} = \beta\kappa_\beta \cos(L_2\beta);
\]
\[a_{46} = \beta\kappa_\beta \sin(L_2\beta); \quad a_{47} = e^{-L_1\alpha}\alpha\kappa_\alpha; \quad a_{48} = e^{-L_1\alpha}\alpha\kappa_\alpha; \quad \text{and} \quad a_{48} = e^{-L_1\alpha}\alpha\kappa_\alpha.
\]

The values of \(\omega\) that satisfy this equation are the natural frequencies of the system.

General formulas of the generalized function coefficients \(C_1\) and \(C_2\) can be determined using the initial conditions and the property of orthogonality of the mode shapes. These formulas are

\[C_1 = I_w \cos \omega_n t_0 - \frac{I_v}{\omega_n} \sin \omega_n t_0\]
\[C_2 = I_w \sin \omega_n t_0 + \frac{I_v}{\omega_n} \cos \omega_n t_0\]

with

\[I_w = \frac{\int_{L_1} w_0\phi + \eta^2\theta_0\psi_n \, dx}{\int_{L_1} \phi_n^2 + \eta^2\psi_n^2 \, dx}\]
\[
I_v = \frac{\int_{-l_0}^{l_0} v_0 \phi_n + \eta^2 \omega_0 \psi_n \, dx}{\int_{-l_0}^{l_0} \phi_n^2 + \eta^2 \psi_n^2 \, dx}
\]

The parameters \( t_0, w_0, v_0, \) are the dimensionless initial time, transverse displacement, cross-section rotation, transverse velocity, and cross-section rotational velocity respectively, and \( \phi_n \) is the \( n \)th dimensionless translational mode shape for the pole, and \( \psi_n \) is the \( n \)th rotational mode shape for the pole.

For the case of rigid body motion (\( \omega = 0 \)), the time variation of the generalized coordinate is modified, as are the coefficients \( C_1 \) and \( C_2 \). In this case the mode shapes \((\phi, \psi)\) are \((x, \dot{x})\) for rotational motion, and the generalized time function will be

\[
q_n = C_1 t + C_2
\]

with

\[
C_1 = I_v, \\
C_2 = I_w - I_v t_0
\]

2.4.9 Upper bound of the initial impact force

Figure 2 shows a schematic of the deflected shape of the system for a very short time after first impact. The time is so short that the effective pole length, \( 2c_i t^* \), is so small that the deformation is dominated by shear. The shear strain in the pole is \( \gamma = \frac{\Delta^*}{c_i t^*} \), in which \( \Delta^* \) is the displacement of the pole at the impact point relative to the as-yet unaffected centerline. Accordingly, the force required to cause the shear deformation is

\[
F^* = 2\kappa G A \gamma = 2\kappa G A \frac{\Delta^*}{c_i t^*}
\]

(2.23)
The force in the contact spring is

\[ F_s^* = k^* \left( \Delta^* - v_0^* \right) \]

which can be rearranged to

\[ \Delta^* = \frac{F_s^*}{k^*} + v_0^* \]  

(2.24)

Substituting Eq. (2.24) into Eq. (2.23) gives

\[ \frac{F^* c_s t^*}{2\kappa GA} = \frac{F_s^*}{k^*} + v_0^* \]  

(2.25)

From equilibrium of forces at the impact point,

\[ F^* = -F_s^* \]  

(2.26)

Substituting Eq. (2.26) into Eq. (2.25) gives

\[ F^* \left( \frac{c_s t^*}{2\kappa GA} + \frac{1}{k^*} \right) = v_0^* \]
and re-arranging terms results in

\[ F^* = \frac{v_0 t^*}{\left( \frac{c_s t^*}{2\kappa GA} + 1 \right)} \]  

(2.27)

By taking the limit of Eq. (2.27) as \( k^* \to \infty \), an upper bound of the initial impact force is

\[ F_{\infty}^* = \frac{2\kappa GA v_0}{c_s} = 2\sqrt{\kappa GA \rho A v_0} = 2\sqrt{k_{sh} m v_0} \]

(2.28)

in which \( k_{sh} = \kappa GA / L^2 \) and \( m = \rho A L^2 \). Note that Eq. (2.28) is exact for rigid stops.

2.4.10 Impact duration

The impact duration for the first impact event is \( t_d = L / c_s \) (impact at the center), which is the time the shear wave in the pole takes to return to the impact point from the free end.

2.4.11 Peak Force using Euler-Bernoulli beam theory

The (time) Laplace transform provides a convenient approach to obtain the limiting impact force for an Euler-Bernoulli beam. Indeed, using the initial value theorem, a simple calculation shows that

\[ F \propto \lim_{s \to \infty} s \frac{dW}{ds} = \lim_{s \to \infty} \frac{-2\sqrt{2} \sqrt{\kappa v_0} e^{s \sqrt{2} \kappa} \left( \sin \left( \sqrt{2} \kappa t \sqrt{s} \right) + \sinh \left( \sqrt{2} \kappa t \sqrt{s} \right) \right)}{e^{2s \sqrt{\kappa \rho A v_0}} + e^{2s \sqrt{\kappa \rho A v_0}} + 4e^{2s \sqrt{\kappa \rho A v_0}} + e^{2s \sqrt{\kappa \rho A v_0}} + 1} = O(\sqrt{s}) \]

(2.29)

where \( s \) is the independent variable in the transformed space. Thus, as \( t \to 0 \), i.e. \( s \to \infty \), the instantaneous impact force is unbounded according to Euler-Bernoulli beam theory. This result is consistent with Eq. (2.28) by letting the shear stiffness become large, given that Euler theory corresponds to infinite shear stiffness.

2.4.12 Mechanical Energy

The energy exchange during impact provides insight on the mechanics of impact force generation and development. The nondimensional mechanical energy of the Timoshenko beam can be written as

22
\[ H = K_w + K_\theta + V_\gamma + V_k \]  \hspace{1cm} (2.30)

where

\[ K_w = \int_{-L}^{0} \frac{1}{2} \left( \frac{\partial w}{\partial t} \right)^2 dx + \int_{0}^{L} \frac{1}{2} \left( \frac{\partial w}{\partial t} \right)^2 dx \]  \hspace{1cm} (2.31)

\[ K_\theta = \int_{-L}^{0} \frac{\eta^2}{2} \left( \frac{\partial \theta}{\partial t} \right)^2 dx + \int_{0}^{L} \frac{\eta^2}{2} \left( \frac{\partial \theta}{\partial t} \right)^2 dx \]  \hspace{1cm} (2.32)

\[ V_\gamma = \int_{-L}^{0} \frac{1}{2} \gamma^2 dx + \int_{0}^{L} \frac{1}{2} \gamma^2 dx \]  \hspace{1cm} (2.33)

\[ V_k = \frac{1}{2} kw^2 (0, t) \]  \hspace{1cm} (2.35)

\[ \gamma = \frac{\partial w}{\partial x} - \theta \] is the section shear strain and \( \kappa_b = \frac{\partial \theta}{\partial x} \) is the beam curvature. This mechanical energy is conserved during the beam motion and the analysis of the exchange of the energy among its components determines the impact-force history.

The exchange of energy is locally dependent on the evolution of the energy density for each kinetic and potential energy component. The governing equations for each energy density can be obtained directly from the model equations and kinetic considerations. Accordingly, let us consider the following form of the Timoshenko model:

\[ \frac{\partial \nu}{\partial t} = \frac{1}{2} \frac{\partial \gamma}{\partial x} - kw \delta \]  \hspace{1cm} (2.36)

\[ \eta^2 \frac{\partial \Omega}{\partial t} = \frac{\partial \kappa_b}{\partial x} + \frac{\gamma}{\tau} \]  \hspace{1cm} (2.37)

where \( \nu = \frac{\partial w}{\partial t} \) and \( \Omega = \frac{\partial \theta}{\partial t} \) denote the transverse velocity and the section angular velocity,
respectively. Also, taking time derivatives of the kinematic relations defining the shear strain and the curvature, and assuming sufficient regularity of the solution to allow for the interchanging of time and space derivatives, we obtain

\[
\frac{\partial \gamma}{\partial t} = \frac{\partial \nu}{\partial x} - \Omega 
\]  
(2.38)

\[
\frac{\partial \kappa_b}{\partial t} = \frac{\partial \Omega}{\partial x} 
\]  
(2.39)

The governing equation for the energy density for each component can be derived by multiplying each of Eqs. (2.36) - (2.39) by its evolving variable. For instance, to obtain the governing equation for \( \kappa_w = \frac{1}{2} \nu^2 \), we multiply Eq. (2.36) by \( \nu \) to obtain

\[
\frac{\partial \kappa_w}{\partial t} = \nu \frac{\partial \gamma}{\partial x} - \frac{\partial V_k}{\partial t} \delta 
\]  
(2.40)

Similarly, we have

\[
\eta^2 \frac{\partial \kappa_b}{\partial t} = \Omega \frac{\partial \kappa_b}{\partial x} + \frac{\Omega}{\tau} 
\]  
(2.41)

\[
\frac{\partial \nu_\gamma}{\partial t} = \gamma \frac{\partial \nu}{\partial x} - \frac{\Omega}{\tau} 
\]  
(2.42)

\[
\frac{\partial \nu_\kappa}{\partial t} = \kappa_b \frac{\partial \Omega}{\partial x} 
\]  
(2.43)

in which \( \kappa_b = \frac{1}{2} \Omega^2 \), \( \nu_\gamma = \frac{1}{2\tau} \gamma^2 \) and \( \nu_\kappa = \frac{1}{2} \kappa_b^2 \). It follows that at the impact point, the spring energy changes at a rate proportional to the jump in the shear power, i.e.,

\[
\frac{dV_k}{dt} = \frac{\nu(0,t)}{\tau} \left[ \gamma(0+,t) - \gamma(0-,t) \right] 
\]  
(2.44)

The energy exchange between the other components is more complex. Away from the origin, \( \kappa_w \) exchanges energy with \( \nu_\gamma \) through the local shear power term
\[
\frac{1}{\tau} \frac{\partial}{\partial x} (\nu \gamma) = \nu \frac{\partial \gamma}{\partial x} + \gamma \frac{\partial \nu}{\partial x}
\]  

(2.45)

This term, \( \frac{1}{\tau} \frac{\partial}{\partial x} (\nu \gamma) \), represents a local source of power from the shear strain which, when integrated, accounts for the power being generated at the beam ends. Besides this exchange with \( \kappa_w \), the shear energy density \( \nu \), also directly trades local specific energy with the rotational kinetic energy \( \nu \) through \( \frac{\gamma \Omega}{\tau} \).

Finally, in addition to the latter exchange, \( \kappa_b \) trades energy with the potential energy density \( \nu \) due to bending through the rotational local source

\[
\frac{1}{\tau} \frac{\partial}{\partial x} (\Omega \kappa_b) = \Omega \frac{\partial \kappa_b}{\partial x} + \kappa_b \frac{\partial \Omega}{\partial x}
\]

(2.46)

An analogous manipulation of the Euler-Bernoulli model equations yields the following equations for the kinetic and potential energy density components:

\[
\frac{\partial \kappa_w}{\partial t} = -\nu \frac{\partial^2 \kappa_E}{\partial x^2} - \frac{dV_k}{dt} \delta
\]

(2.47)

\[
\frac{\partial \nu}{\partial t} = \kappa_E \frac{\partial^2 \nu}{\partial x^2}
\]

(2.48)

in which \( \kappa_w = \frac{1}{2} \nu^2 \), \( \nu = \frac{1}{2} \kappa_E^2 \), and \( \kappa_E = \frac{\partial^2 w}{\partial x^2} \). Moreover, we have

\[
\frac{dV_k}{dt} = -\nu(0, t) \left[ \frac{\partial \kappa_E}{\partial x} \bigg|_{(0, t)} - \frac{\partial \kappa_E}{\partial x} \bigg|_{(0, -t)} \right]
\]

(2.49)

2.5 Results

As stated previously, the primary motivation for this work is to determine the force from debris impact. It is interesting to investigate how the impact force varies with practical values of the dimensionless parameters. Regardless of where along the beam impact occurs the impact
force time history is the same until the waves have time to return from the end of the shorter segment. In addition, the force time history is independent of the initial angular velocity because the impact duration is very small and the beam rotation is negligible within the impact duration. As a result, we focus on impact at midspan, which would seem to lead to longer contact durations, and on zero initial angular velocity.

The pole has a weight of 342 kg (754 lb), length of 9.14 m (30 ft), and a diameter of 0.305 m (1 ft). This is somewhat less than the prototypical 1,000 lb wood pole suggested by a common U.S. design guideline (ASCE 2010). Therefore, for the parameter study the beam has a slenderness ratio $\lambda_b = 120$, $\eta = 1$, and a shear coefficient $\kappa = 0.9$, which is commonly used for a solid circular section (although more accurate estimates are available (Cowper 1966)). For an isotropic material, $\tau = 2(1+\nu)/\kappa$, in which $\nu$ is the Poisson ratio. However, wood is not an isotropic material, and wood properties vary widely. We adopt here a value of $\tau = 10$ for our prototypical beam, which is at the low end of values suggested in Yoshihara et al. (1998).

Note that the dimensional force at time zero for rigid impact is $F_0 = 2\sqrt{\kappa G A\rho}$. Based on $G = 0.89$ GPa (129 ksi) given in Yoshihara et al. (1998) for spruce, the estimated rigid impact force for our prototypical beam impacting at a physically reasonable 10 km/h (6.2 mph) is $F_0 \approx 300$ kN (67,000 lb) with an approximate duration of 8 ms (milliseconcds).

Unless otherwise stated below, Timoshenko beam theory is used. Results for Euler-Bernoulli beam theory will be compared subsequently. Also, in the following, for convenience, the impact force is non-dimensionalized by the theoretical impact force based on rigid impact given by Eq. (2.28), $F/\bar{F}_0$, time is nondimensionalized as $\bar{t} = t^*/t_d^* = t^* c_s/\bar{L}^*$ and nondimensional space is defined as $\bar{x} = x^*/\bar{L}^*$. For impact at midspan of the beam, $t_d^* = \bar{L}^*/c_s$, which is the time it takes a shear wave to propagate to the free end and back to the point of impact. It is a measure of the approximate duration of impact.

2.5.1 Effect of spring stiffness

First, we consider a range of $\bar{k} = k^*/(\kappa G A/\bar{L}^*)$, which is the ratio of the spring stiffness $k^*$ to a measure of the shear stiffness of the beam. A convergence study was carried out to
determine what value of $\bar{k}$ is sufficiently close to rigid impact. As shown in Figure 3, there is only a modest difference between $\bar{k} = 1200$ and 1800, and $\bar{k} = 1200$ is chosen to represent rigid impact for the current study. However, even $\bar{k} = 600$ is quite stiff, and the primary difference is a slightly lower initial impact force and a slightly longer rise time. We note that contact durations are all less than 1 for this range of $\bar{k}$.

(a) Impact force over entire duration
(b) Close up of initial peak impact force

Figure 3: Impact force for approximately "rigid" impact with $\tau = 10, \eta^2 = 1, \lambda_b = 120$

The value of $\bar{k} = 1$ is taken as the smallest stiffness considered. Figure 4 and Figure 5 show the impact force for different values of $\bar{k}$ and two values of $\tau$. (As explained subsequently, $\tau = 20/9$ represents a reasonable lower bound.) For very soft springs, the response resembles the response of a single-degree-of-freedom oscillator. However, this quickly changes as $\bar{k}$ increases. For all cases with $\bar{k} \geq 10$, the response is qualitatively similar, with the initial peak impact force increasing toward the limit of 1 as $\bar{k}$ increases. The second peak in the force also increases, and can exceed one for large spring stiffnesses and $\tau = 10$. There are also high frequency oscillations that start near the end of contact.
Figure 4: Impact force for a range of $k$, with $\eta^2 = 1$, $\lambda_B = 120$ and $\tau = 10$

Figure 5: Impact force for a range of $k$, with $\eta^2 = 1$, $\lambda_B = 120$ and $\tau = 20/9$

For stiff springs it can be seen that the force drops off quickly from the initial peak. The initial response is dominated by localized shear behavior around the point of impact, and the
force decreases as flexure begins to occur and strain energy is converted from shear strain energy to bending strain energy.

2.5.2 Effect of $\tau$

As mentioned previously, for an isotropic beam $\tau = 2(1 + v) / \kappa$. It is interesting to see how the impact force varies based on different Poisson’s ratio and $\kappa$. For $0 \leq v \leq 0.5$ and $\kappa = 0.5$ (circular or square tube), $5/6$ (rectangular cross section) and $0.9$ (solid circular cross section), values $\tau = 20/9, 3, 4, 5,$ and $6$ are considered. The value $20/9$ corresponds to $v = 0$ and $\kappa = 0.9$, and therefore represents a reasonable lower bound. The value $6$ corresponds to $v = 0.5$ and $\kappa = 0.5$, and therefore represents a reasonable upper bound for an isotropic beam of nearly incompressible material. The impact force time histories are shown in Figure 6. The dependence of the impact force on $\tau$ does not appear to be as strong as the dependence on spring stiffness. A consistent trend is that the initial peak (nondimensional) impact force increases with increasing $\tau$. However, even for the range considered, the minimum peak is still greater than $0.88$. At the end of contact, the variation is larger. In general, the second peak increases with increasing $\tau$, although not consistently. For example, the second peak for $\tau = 5$ is about $0.9$, while the second peak for $\tau = 6$ is only about $0.4$. 

30
Figure 6: Impact force for a range of $\tau$, with $k = 1200, \eta^2 = 1, \lambda_B = 120$

### 2.5.3 Effect of slenderness ratio $\lambda_B$

The effect of slenderness ratio is shown in Figure 7 for nearly rigid impact where $k = 1200$ and Figure 8 for a soft spring where $k = 10$. One interpretation for these plots is to assume that $L^*$ remains constant, so that the horizontal axis represents the same point in time for all cases, and let $r$ vary. It can be seen then that as the radius of gyration increases (bending stiffness increases relative to shear stiffness), the importance of the shear deformation increases and the drop in impact force is slower.
Figure 7: Impact force for a range of $\lambda_B$, with $\bar{k} = 1200, \eta^2 = 1, \tau = 10$

Figure 8: Impact force for a range of $\lambda_B$, with $\bar{k} = 10, \eta^2 = 1, \tau = 10$
2.5.4 Effect of $\eta$

The effect of mass moment of inertia is shown in Figure 9. As can be seen, the moment of inertia has a negligible effect on the impact force for the soft spring case. In the case of nearly rigid impact, its effect on the whole trend of the impact force-time curve is negligible except for some oscillations near the end of the impact. This conclusion is interesting, because some structural finite element codes use a lumped mass matrix that includes translational masses but not rotational inertias.

![Figure 9: Impact force for $L/r = 120, \tau = 10$ with soft and stiff springs and $\eta^2 = 1$ and $\eta^2 = 0.01$](image_url)

2.5.5 Asymmetric impact of rotating beam

The focus thus far has been on the impact at midspan, which would seem to lead to longer contact durations, and on zero initial angular velocity. Figure 10 shows the impact force history for an asymmetric impact at quarter span for different initial angular velocities. As before, the impact force in this plot is nondimensionalized by the force for the impact on a rigid structure (Eq. (2.28)), but here using the velocity at the impact point when the beam is rotating. The plot
shows that the initial impact dynamics is dominated by the vibrational motion in the immediate neighborhood of the impact. So regardless of where along the beam impact occurs, the impact force-time history is the same until the waves have time to return from the end of the shorter segment. In addition, the time history of the impact force is independent of the initial angular velocity because the impact duration is very small and the variation in the beam rotation is negligible within the impact duration. The main effect of the asymmetric impact is in lowering the second peak of the impact force because of the asynchrony of returning waves. Although not shown, it should be remarked that the vibrational motion of the beam depends on the position of the impact as well as on the initial rotation. For instance, the dependence of vibration history on the impact location is evident in the dispersed waves: the second force peak is broader and is preceded by ripples of lower wave number for asymmetric impact when compared with the symmetric impact.

Figure 10: Impact force for impact at quarter span, \( \lambda_B = 120 \), \( \tau = 10 \), \( \eta^2 = 1 \) and \( \bar{k} = 1200 \)

(All plots coincide)
2.5.6 Euler vs. Timoshenko

The previous results are for Timoshenko beam theory. It is of interest to determine for which parameter regimes Euler-Bernoulli beam theory gives results similar to Timoshenko beam theory. Qualitatively, we would expect that for a large slenderness ratio such as $\lambda_b = 120$, Euler would give reasonably good results. However, as shown previously, as the spring stiffness increases, the Euler impact force grows without bounds. Figure 12 shows the nondimensional time-force relations for different values of $\vec{k}$. It can be seen that for a soft spring stiffness in which $\vec{k} = 1$, Euler-Bernoulli beam theory gives close results to Timoshenko beam theory, because the shear force is relatively small and the impact is absorbed by the bending action of the beam and the spring. On the other hand, for large $\vec{k}$, the impact force increases unboundedly using Euler-Bernoulli beam theory. For instance, for the case of $\vec{k} = 1200$, the force calculated using the Euler-Bernoulli beam theory is almost three times the force calculated using the Timoshenko beam theory, and the duration is much shorter. The impact force is larger in the Euler-Bernoulli beam model because the main mechanism of energy transfer at the start of impact, namely, the transfer of energy from the translational kinetic energy to shear strain potential energy, is lacking in the Euler model. This shear strain is replaced by an unphysical flexural strain in the Euler-Bernoulli model that over predicts the impact force. The shorter duration, however, is due to differences in dispersion at short and long time scales and in particular it can affect the duration of the impact.

As it is well known (see e.g. H.Kolsky (1963), Y.C.Fung (1965) and S.H.Crandall (1968)), the group speed increases without bound in the Euler-Bernoulli model, whereas the Timoshenko model predicts two different branches for the group speed. As the number increases, one of the modes asymptotically converges toward the shear speed, while the second mode converges toward the longitudinal wave speed (for convenience, we plot the group speeds for the different models in Figure 11). Thus, during impact against a stiff spring, a broadband wave whose wave groups at the high wave-number end of the spectrum can move much faster in the Euler-Bernoulli model than in the Timoshenko beam. These distinct dispersion characteristics of the two beam models explain why the nondimensional time duration is close to one for the impact on a stiff spring for the Timoshenko model, and why the impact ends prematurely in the Euler-
Bernoulli model under the same conditions. Indeed, because of the unbounded nature of the group speed in the Euler-Bernoulli model, the trend in this model is for decreasingly shorter duration as stiffness of the spring increases.

Figure 11: Branches of dimensionless group speed $\bar{c}_g = c_g^*/c_s$ as a function of the dimensionless wave number $\xi = \xi^* \tau$ for $\tau = 10$
(a) $\bar{k} = 1$
b) $\bar{k} = 5$
\( \overline{k} = 10 \)
d) $\bar{k} = 100$
2.5.7 Energy breakdown and beam vibration

The energy exchange gives an insight on the beam vibration and in particular into generation and behavior of the impact force. Figure 13 shows the time evolution for the potential and kinetic energy components. It should be noted that the results are for a stiff spring, where the maximum impact force is dominated by the shear wave and the spring energy is very small: theoretically, for rigid impact, the spring energy vanishes. The maximum impact force occurs when the spring is at maximum compression and the speed of the point of contact is zero. The initial peak in the force is dominated by shear and after the initial impact the force is dominated by the exchange between translational kinetic energy and potential bending energy. Indeed, without rotation or pre-strain, all initial energy is translational kinetic energy. Locally, the kinetic energy density, and by integration the kinetic energy, exchange energy with the spring and shear potential energy.

e) $\bar{k} = 1200$

Figure 12: Comparison of impact forces between Timoshenko and Euler-Bernoulli beam theories for different $\bar{k}$, with $\lambda_B = 120, \tau = 10, \eta^2 = 1$
(Eq. (2.40) and Eq. (2.41)). Thus, initially, the beam’s initial energy is transformed into potential shear energy and potential spring energy. This explains the sharp rise in the shear and spring energies and the dominance of shear at the initial impact. Potential bending energy, on the other hand, exchanges energy with kinetic rotational energy and depends on the build-up of curvature and rotation (Eq. (2.42) and Eq. (2.43)). As such it is slower to respond to the impact; see Figure 13b. In the Euler-Bernoulli model, the absence of the neglected shear strain causes the kinetic energy to exchange energy directly with flexural strain (Eq.(2.47)). The results show an unrealistic increase in the impact peak force; see Figure 12.

(a) All energy components over the entire impact duration
Figure 13: Time evolution of potential and kinetic energy components (log scale) for $\lambda_B = 120$, $\tau = 10, \eta^2 = 1$ and $\bar{k} = 1200$

Figure 14 shows the potential energy densities as a function of space and time. It is apparent that at the time of impact, there is a large spike in the shear strain energy as the impact is absorbed via shearing and spring deformation. This is then quickly transferred to bending strain energy around the impact as curvature is established and the Euler-Bernoulli hypothesis of zero shear for low wave number is approached. The shear wave propagates down the bar until it is reflected at around $t^*_a / 2 = L / 2c_s$ ($t = 1/2$). Of interest is the ensuing motion after reflection, which displays two modes of dispersion: one corresponding to the flexural mode and traveling at group speed $c_g \approx 1$ and a second, shear, quicker mode, at group speed $c_g \approx \sqrt{\tau}$. The second mode is not as distinct as the dispersion of the flexural waves that is noticeable in Figure 14, but is clearly apparent in the dispersion of the rotational kinetic energy in Figure 15. Also evident in this figure is a peak in the rotational kinetic energy, or “whipping” effect, as the latter is reflected. The high frequency oscillation at the center of the beam near the end of impact explains the high

(b) Close up of all energy components around the initial peak force
harmonics of the impact force before separation. Note that this second mode of dispersion is fully absent in the Euler-Bernoulli model; see Figure 12.

Figure 14: Space-time plot of potential energy density for $\lambda_B = 120, \tau = 10, \eta^2 = 1$ and $\bar{k} = 1200$
(a) Translational kinetic energy density

(b) Rotational kinetic energy density

Figure 15: Space-time plot of kinetic energy density for $\lambda_b = 120$, $\tau = 10$, $\eta = 1$, $k = 1200$

2.6 Summary

An analytical solution for the linear response of a Timoshenko beam impacting a stop modeled as a spring has been presented. The solution, which is based on modal superposition,
admits contact at an arbitrary point within the beam span, as well as beam rotational as well as translational initial velocities. As such, the present solution is a generalization as compared to previous solutions. It has been shown that the initial impact is dominated by shear behavior, and that for stiff stops, Euler-Bernoulli beam theory can significantly overestimate the initial impact force and underestimate the contact duration. Indeed, based on Euler-Bernoulli beam theory the impact force grows without bounds as the stop stiffness increases, whereas Timoshenko beam theory shows the impact force is bounded. As a general rule, it is recommended that Timoshenko beam theory rather than Euler theory be used for impact studies. Parameter studies have shown that the impact force time history is controlled primarily by the ratio of the stop stiffness to the beam shear stiffness. For homogeneous beams, the sectional rotational inertia has little effect on the impact forces.
Chapter 3
Beam Response to Longitudinal Impact by a Pole

3.1 Introduction

This chapter investigates a pole striking a column longitudinally. The model involves a beam governed by Timoshenko beam theory and a pole governed by the one-dimensional wave equation. The discussion is restricted to elastic impact and linear response because of the low speeds involved in debris impact. For the case of a free-free beam hitting a stop, the beam, which has initial translational and rotational velocities, can strike the stop anywhere along the beam. The stop is flexible and it is modeled as a rod with wave propagation. For the case of axial impact of a pole against a column, any combination of column end conditions are allowed, and again contact can occur anywhere along the column. Although the pole-column impact is of primary interest, the model is also valid for the case of a free-free beam, which has initial translational and rotational velocities, hitting a stop anywhere along the beam. The stop is flexible and it is modeled as a rod with wave propagation.

The performance of Euler beam theory is again investigated in this chapter. In addition, the effect on the impact force of ignoring the rotary inertia in Timoshenko beam theory is investigated; this has practical significance because some finite element codes include shear deformation but ignore rotary inertia. Finally, a detailed discussion of the transfer of energy between kinetic energies and the potential (strain) energies is provided. The analysis of energy exchange examines the contributions of the different energy components to the impact dynamics.

This chapter is organized as follows. First, the governing equations and the chosen nondimensionalization of the variables are presented. Then the analytical solutions, which are based on superposition of the normal modes of vibration, are developed. Results of a study on impact response for parameters within a range that might be seen in debris impact are presented and interpreted. The final section contains conclusions from the present study.
3.2 Physical System

A schematic of the system is shown in Figure 16. The figure shows a pole with an initial velocity striking a column. A spring between the pole and the column represents a local contact stiffness, $k_y$. In this scenario, the column would likely have displacement boundary conditions (e.g., fixed-fixed or pinned-pinned) and would have zero initial velocity. In a second scenario, the beam could be traveling with an initial velocity $v_0$ (at $x_i = 0$) and angular rotation $\omega_0$ and impacts a ‘pole’ or flexible ‘stop’. The pole may have a zero-displacement boundary condition at the right end. In both cases, the origin of the coordinates $x_i - x_2$ is located at the point of impact. For generality of the discussion, the beam in the second scenario also will be referred to herein as a column, unless the discussion is specific to a translating beam.

The column and the pole are assumed to be elastic and homogeneous, with constant cross-sectional properties. The pole has Young’s modulus $E_p$; sectional area $A_p$; mass density $\rho_p$; and length $L_p$. The column is modeled by Timoshenko beam theory. It has Young’s modulus $E$; shear modulus $G$; mass density $\rho$; sectional area moment of inertia $I$; sectional area $A$; sectional rotary inertia $I_m = \eta^2 \rho I$, where $\eta^2$ is defined as the ratio of $I_m$ to $\rho I$; shear coefficient $\kappa$; and total length $L = L_1 + L_2$.

For these systems, the fundamental kinematic variables, which are functions of time $t$, are the axial displacement in the pole, $u(x_i, t)$, and the transverse displacement $w(x_i, t)$ and cross-sectional rotation $\theta(x_i, t)$ of the column. The latter is taken as clockwise positive.

It is convenient to work with dimensionless variables. For Timoshenko beam theory it is common to non-dimensionalize the length scale by $r = \sqrt{I/A}$ and time by $c_0/r$, where $c_0 = \sqrt{E/\rho}$ (Boley and Chao 1955; Xing et al. 2002). Adopting this for the entire system results in the following non-dimensional variables:

$$\bar{x}_i = \frac{x_i}{r} ; \bar{w} = \frac{w}{r} ; \bar{x}_2 = \frac{x_2}{r} ; \bar{u} = \frac{u}{r} ;$$

and

$$\bar{T} = t \frac{c_0}{r} \quad \text{(3.1)}$$
Additional non-dimensional variables include

\[ l_i = \frac{L_i}{r}, i = 1, 2; \quad l_p = \frac{L_p}{r}; \quad \bar{k} = \frac{k_i}{E A / r}; \quad \tau = \frac{E}{\kappa G} = \frac{c_0^2}{c_i^2}; \quad \mu = \frac{c_{0p}}{c_0} \]  \hspace{1cm} (3.2)

in which \( c_s = \sqrt{\kappa G / \rho} \) and \( c_{0p} = \sqrt{E_p / \rho_p} \).

Figure 16: Schematic of beam/column and pole

### 3.3 Equation of Motion

The equations of motion for the column are

\[
\frac{1}{\tau} \left( \frac{\partial^3 \bar{w}}{\partial x_i^3} - \frac{\partial \bar{\theta}}{\partial x_i} \right) - \bar{k} \left[ \bar{w}(0, \tau) - \bar{w}(0, \tau) \right] \delta - \frac{\partial^2 \bar{w}}{\partial \tau^2} = 0
\]  \hspace{1cm} (3.3)
\[
\frac{\partial^2 \theta}{\partial x_i^2} + \frac{1}{\tau} \left( \frac{\partial \bar{w}}{\partial x_i} - \theta \right) - \eta^2 \frac{\partial^2 \theta}{\partial t^2} = 0
\]  

(3.4)

\( \delta \) in Eq. (3.3) is the Dirac delta. Note that the positive definition of shear strain, \( \gamma = \left( \frac{\partial \bar{w}}{\partial x_i} - \theta \right) \), is such that positive shear force acts in the \(+x_2\) direction on a positive \(x_1\) face. To deal with the discontinuity in shear at \( x_i = 0 \), let \( \bar{w} = \bar{w}^+ \) for \( x_i \geq 0 \) and \( \bar{w} = \bar{w}^- \) for \( x_i \leq 0 \), and similarly for \( \theta \).

The equation of motion for the pole is the one-dimensional wave equation:

\[
\frac{\partial^2 \bar{u}}{\partial x_2^2} - \frac{1}{\mu^2} \frac{\partial^2 \bar{u}}{\partial t^2} = 0
\]

(3.5)

As mentioned, a number of different boundary conditions are possible. For the column these are either zero displacement or shear and zero rotation or moment at \( x_i = l_1 \) and \( x_i = l_2 \), and for the pole they are zero displacement or zero force at \( x_2 = l_p \). Initial displacements are zero and options for initial velocities are as described earlier.

The compatibility equations at \( x_i = 0 \) are equal displacement, rotation, and curvature in the column, as well as the jump condition for shear

\[
\left( \frac{\partial \bar{w}'}{\partial x_i} - \theta' \right) - \left( \frac{\partial \bar{w}}{\partial x_i} - \theta \right) - \bar{k} \tau (\bar{w} - \bar{u}) = 0
\]

(3.6)

The compatibility between the pole and the column results in

\[
\frac{\partial \bar{u}}{\partial x_2} - \frac{1}{\alpha_m \mu^2} \bar{k} \left( \bar{w} - \bar{u} \right) = 0
\]

(3.7)

in which \( \alpha_m = \rho_p A_p / \rho A \).
3.4 Analytical Solution

3.4.1 Mode Shapes

A standard approach of separation of variables is adopted. The column mode shapes on each side of the discontinuity at \( x_i = 0 \) are the usual Timoshenko beam mode shapes, i.e.,

\[
\left\{ \phi^+ \left( x_i \right) \right\} = \begin{bmatrix} \phi_1^+ \cos \beta x_i + \phi_2^+ \sin \beta x_i + \phi_3^+ e^{\alpha x_i} + \phi_4^+ e^{-\alpha x_i} \\ \psi^+ \left( x_i \right) \end{bmatrix}
\]

\[
\left\{ \phi^- \left( x_i \right) \right\} = \begin{bmatrix} \kappa_{\beta} \left( \phi_1^- \sin \beta x_i - \phi_2^- \cos \beta x_i \right) + \kappa_{\alpha} \left( \phi_3^- e^{\alpha x_i} - \phi_4^- e^{-\alpha x_i} \right) \\ \psi^- \left( x_i \right) \end{bmatrix}
\]

with

\[
\beta = \sqrt{\frac{\omega^2 (\eta^2 + \tau) + \omega^2 (\eta^2 - \tau)^2 + 4}{2}}, \quad \alpha = \sqrt{\frac{-\omega^2 (\eta^2 + \tau) + \omega^2 (\eta^2 - \tau)^2 + 4}{2}}
\]

\[
\kappa_{\beta} = \frac{(\tau \omega^2 - \beta^2)}{\beta}, \quad \kappa_{\alpha} = \frac{(\tau \omega^2 + \alpha^2)}{\alpha}
\]

and \( \omega \) is the nondimensional frequency. Note that \( \beta \) is always real, but \( \alpha \) may be real or imaginary. Two special mode shapes must be considered. The first are the rigid body modes (if the boundary conditions allow such) when \( w = 0 \). The second is the special case of \( \omega \to \frac{1}{(\eta \sqrt{\tau})} \), \( \tau \to 0 \), in which the mode shapes can be obtained by taking the limit of Eq. (3.8). These eigenfunctions are consistent with Xing et al. (2002), who considered only the case \( \eta = 1 \). For the pole, the modes shapes are \( \phi_p = C_1 \cos \xi x_i + C_2 \sin \xi x_i \) where \( \xi = \omega / \mu \).

3.4.2 Characteristic Equation

The natural frequencies and the coefficients of the mode shapes are determined by imposing the five compatibility conditions and five boundary conditions. The ten equations result in a system of equations that determines the natural frequencies \( \omega \). For the case of a free-free column and fixed end pole, the characteristic equation is obtained from the determinant:
with \( a_{15} = (\beta + \kappa_\beta) \sin \beta l_1 \); \( a_{16} = (\beta + \kappa_\beta) \cos \beta l_1 \); \( a_{17} = (\alpha - \kappa_\alpha) \exp(-\alpha l_1) \); \( a_{18} = -(\alpha + \kappa_\alpha) \exp(\alpha l_1) \); \( a_{21} = -(\beta + \kappa_\beta) \sin \beta l_2 \); \( a_{22} = (\beta + \kappa_\beta) \cos \beta l_2 \); \( a_{23} = (\alpha - \kappa_\alpha) \exp(-\alpha l_2) \); \( a_{24} = -(\alpha + \kappa_\alpha) \exp(-\alpha l_2) \); \( a_{35} = \beta \kappa_\beta \cos \beta l_1 \); \( a_{36} = -\beta \kappa_\beta \sin \beta l_1 \); \( a_{37} = \alpha \kappa_\alpha \exp(-\alpha l_1) \); \( a_{38} = \alpha \kappa_\alpha \exp(\alpha l_1) \); \( a_{41} = \beta \kappa_\beta \cos \beta l_2 \); \( a_{42} = \beta \kappa_\beta \sin \beta l_2 \); \( a_{43} = \alpha \kappa_\alpha \exp(\alpha l_2) \); and \( a_{44} = \alpha \kappa_\alpha \exp(-\alpha l_2) \). For other boundary conditions, rows 1, 2, 3, 4, and 9 in the determinant must be modified accordingly. As noted earlier, \( \alpha \) can be either real or complex. In the latter case, the characteristic equation is also complex.

Mathematica (Wolfram 2010) was used to obtain the natural frequencies herein. Starting from zero, a very small increment in \( \omega \) was used to search for a segment in which a root existed, and then Newton-Raphson was used to find the root. As an example, an increment of 1/100,000
was used to find 1000 modes. It was observed that if a mode is missed, the force-time curve is not smooth and local peaks and valleys appear along the curve, and if insufficient modes are used, sharp spikes appear around the initial peak of the force.

Once the natural frequencies and modes are determined, the response for initial velocity conditions is obtained as

\[
\begin{bmatrix}
\bar{w}(\bar{x}_1, \bar{T}) \\
\theta^z(\bar{x}_1, \bar{T}) \\
\bar{u}(\bar{x}_2, \bar{T})
\end{bmatrix} = \sum_{n=1}^{p} B_n \begin{bmatrix}
\phi_n^z(\bar{x}_1) \\
\psi_n^z(\bar{x}_1) \\
\phi_{pn}(\bar{x}_2)
\end{bmatrix} \sin \omega_n \bar{T}
\]

(3.9)

given that

\[
B_n = \frac{\int_{-l_1}^{l_2} \left( \phi_n^z v_0 + \eta \psi_n^z \omega_0 \right) d\bar{x}_1 + \alpha_n \int_{0}^{l_2} \phi_{pn} v_{0p} d\bar{x}_2}{\omega_n \int_{-l_1}^{l_2} \left( \phi_n^2 + \eta \psi_n^2 \right) d\bar{x}_1 + \alpha_n \int_{0}^{l_2} \phi_{pn}^2 d\bar{x}_2}
\]

(3.10)
in which \( v_0, \omega_0, \) and \( v_{0p} \) are the initial translational velocity of the column, the initial rotational velocity of the column, and the initial translational velocity of the pole, as appropriate. For example, a stationary column can be hit by a translating pole, or a fixed or free pole can be hit by a translating and rotating beam (except only one dimensional motion of the pole is considered).

Note that the above equations related to the column can be easily modified for a simplified model of a beam hitting a massless stop, i.e., a massless spring, by dropping the terms containing \( u \) and \( \phi_p \).

3.4.3 Initial Peak Impact Force

The initial peak impact force can be estimated as follows. Figure 17 shows a schematic of the deflected shape of the system at time \( t (t << 1) \) after impact. The time is sufficiently small such that the effective beam length, \( 2c_j t \), is so small that the deformation is dominated by shear. The shear strain in the column is \( \gamma = \Delta_2 / c_j t \), in which \( \Delta_2 \) is the displacement of the column at the impact point relative to the as-yet unaffected centerline. Accordingly, the force required to cause
the shear deformation is \( F_c = 2kGA\gamma \). Similarly, the effective length of the pole is \( c_{op}t \), and the strain is \( \varepsilon_p = (\Delta_1 - v_0 t) / c_{op}t \), in which \( \Delta_1 \) is the displacement of the impacted end of the pole. The corresponding force is \( F_p = E_p A_p \varepsilon_p \). The force in the contact spring is just \( F_s = k_s (\Delta_2 - \Delta_1) \). Equilibrium of the massless spring requires \( F_c = F_p = -F_s \), resulting in the impact force

\[
F_c = F_p = \frac{F_{pR} F_{pR}}{F_{pR} + \frac{F_{cR} F_{pK} c_{op}}{v_0^2 k_s t}}
\]

Figure 17: Impact zone at time \( t \)

Two known special cases can be recovered easily from Eq. (3.11). The instantaneous impact force of a pole hitting a rigid column, \( F_c = F_{pR} \), is obtained from Eq.(3.11) by taking the limit as \( k_s \to \infty \) and \( GA \to \infty \). This force can also be written as (Paczkowski et al. 2012)

\[
F_{pR} = \sqrt{k_p m_p v_0}
\]
in which \( k_p = E_p A_p / L_p \) and \( m_p = \rho A_p L_p \). The practical applicability of Eq.(3.12) for debris impact has been shown based on full-scale tests of logs and shipping containers impacting a stiff target (Piran Aghl et al. 2014). Similarly, the instantaneous impact force of a beam hitting a rigid stop, \( F_e = F_{eR} \) (Riggs et al. 2013; Xing et al. 2002) is obtained from the limit as \( k_s \to \infty \) and \( E_p A_p \to \infty \). This force can also be written as

\[
F_{eR} = 2\sqrt{k_{sh}mv_0}
\]

(3.13)

in which \( k_{sh} = \kappa GA / L \) and \( m = \rho AL \). Eqs. (3.12) and (3.13) are exact for rigid stops; note also the similarity in their forms. These forms involving stiffness and total mass may be most convenient for structural designers.

Because of uncertainty in the local contact stiffness, in this study \( k_s \) is taken to be very large. Thus, \( k_s \) can be interpreted as a penalty parameter to couple the beam and the column. The large but finite stiffness \( k_s \) produces a steep ramp at impact with slope proportional to its magnitude, instead of a jump in the force associated with an impact with infinite \( k_s \). This ramp is not only physically more realistic, but it also prevents the occurrence of the Gibbs phenomenon that would otherwise occur with a modal superposition solution for an infinite \( k_s \).

The limit of Eq. (3.11) as \( k_s \to \infty \) results in the instantaneous peak impact force

\[
F_0 = \frac{F_{eR}F_{pR}}{F_{eR} + F_{pR}}
\]

(3.14)

which can also be written as

\[
F_0 = \frac{1}{\alpha_1} \sqrt{k_p m_p v_o} = \frac{2}{\alpha_2} \sqrt{k_{sh} mv_o}
\]

(3.15)

in which \( \alpha_1 = 1 + \alpha_m \mu \sqrt{\tau} / 2 \), and \( \alpha_2 = 1 + 2 / (\alpha_m \mu \sqrt{\tau}) \). In the results section, the impact forces are nondimensionalized by the values predicted by Eq.(3.15), which will demonstrate its applicability. Note that if \( \alpha_1 \) is close to 1, the peak impact force is dominated by the axial response of the pole, whereas when \( \alpha_2 \) is close to 1, the force is dominated by the shear response.
of the column.

Eq. (3.14) also applies to the impact of two orthogonal beams if $F_{ph}$ is replaced by $F_{cr}$ of the second beam. Eq.(3.14) has the same form as the formula for longitudinal impact of two bars, although the derivation here is quite different (Graff 1991; Harrison and Nettleton 1997).

3.4.4 Mechanical Energy

The energy exchange during impact provides insight on the mechanics of impact force generation and development. To simplify notation, $v = \partial w / \partial \tau$ and $\Omega = \partial \theta / \partial \tau$ are the transverse velocity and the section angular velocity of the column, respectively; $v_p = \partial u / \partial \tau$ is the velocity in the pole; $\kappa_b = \partial \theta / \partial \alpha$ is the beam curvature, and $\varepsilon_p = \partial u / \partial \alpha$ is the pole strain. The nondimensional energy densities (per unit length) are defined as follows: $K_v = v^2 / 2$ (column translational kinetic); $K_{\theta} = \eta^2 \Omega^2 / 2$ (rotational kinetic); $K_u = \alpha_m v_p^2 / 2$ (pole translational kinetic); $Y_{\gamma} = \gamma^2 / 2 \tau$ (shear); $Y_k = \kappa_b^2 / 2$ (bending) and $Y_{\varepsilon} = \alpha_m \kappa_b^2 \varepsilon_p^2 / 2$ (pole strain). The spatial integration of the energy densities results in the total energy for each component. The sum of these six components plus the potential energy in the contact spring results in the total nondimensional mechanical energy. Note that the energies are nondimensionalized by $(EAr)$. The model does not include any energy dissipation mechanisms and therefore the mechanical energy is conserved during impact.

The exchange of energy is locally dependent on the evolution of the energy density for each kinetic and potential energy component. The governing equations for each energy density can be obtained directly from the model equations and kinematic considerations. For that purpose, consider the following form of the Timoshenko model:

$$\frac{\partial v}{\partial \tau} = \frac{1}{\tau} \frac{\partial \gamma}{\partial \alpha} - k (\bar{w} - \bar{u}) \delta \quad \text{and} \quad \eta \frac{\partial \Omega}{\partial \tau} = \frac{\partial \kappa_b}{\partial \alpha} + \frac{\gamma}{\tau}$$

(3.16)

Also, taking time derivatives of the kinematic relations for the shear strain and the curvature, and assuming sufficient regularity of the solution to allow for the interchanging of time and space derivatives, one obtains
\[
\frac{\partial \gamma}{\partial t} = \frac{\partial v}{\partial x} - \Omega \quad \text{and} \quad \frac{\partial \kappa}{\partial t} = \frac{\partial \Omega}{\partial x}
\]  

(3.17)

The governing equation for the energy density for each component can be derived by multiplying each of Eqs. (3.16)–(3.17) by its evolving variable. For instance, to obtain the governing equation for the translational kinetic energy density \(K_w\), multiply the first of Eq. (3.16) by \(v\) to obtain

\[
\frac{\partial K_w}{\partial t} = \frac{v \partial \gamma}{\tau \partial x} - \bar{k} (\bar{w} - \bar{u}) v \delta
\]  

(3.18)

Similarly,

\[
\frac{\partial K_{\gamma}}{\partial t} = \Omega \frac{\partial \kappa}{\partial x} + \frac{\gamma \Omega}{\tau} \quad \text{and} \quad \frac{\partial K_{\nu}}{\partial t} = \nu \alpha \mu^2 \frac{\partial \varepsilon_{\nu}}{\partial x}
\]  

(3.19)

\[
\frac{\partial V_{\gamma}}{\partial t} = \frac{\gamma \partial v}{\tau \partial x} - \frac{\gamma \Omega}{\tau} \quad \text{and} \quad \frac{\partial V_{\nu}}{\partial t} = \kappa \frac{\partial \Omega}{\partial x}
\]  

(3.20)

\[
\frac{\partial V_{\nu}}{\partial t} = \alpha \mu^2 \varepsilon_{\nu} \frac{\partial V_{\nu}}{\partial x}
\]  

(3.21)

From equilibrium at the contact point, \(\bar{k} (\bar{w}(0, \bar{r}) - \bar{u}(0, \bar{r})) = \frac{1}{\tau} \left[ \gamma(0^+, \bar{r}) - \gamma(0^-, \bar{r}) \right]\), and it follows that the spring energy changes at a rate proportional to the jump in the shear power:

\[
\frac{dV_{k}}{dt} = \frac{\nu(0, \bar{r}) - \nu(0, \bar{r})}{\tau} \left[ \gamma(0^+, \bar{r}) - \gamma(0^-, \bar{r}) \right]
\]  

(3.22)

The energy exchange between the other components is more complex. Away from the origin, \(K_w\) exchanges energy with \(V_{\gamma}\) through the local shear power term

\[
\frac{\partial K_w}{\partial t} + \frac{\partial V_{\gamma}}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial x} (v \gamma) + \nu \left( \frac{\partial v}{\partial t} - \frac{1}{\tau} \frac{\partial \gamma}{\partial x} \right) - \frac{\gamma \Omega}{\tau} \frac{dV_{k}}{dt} \delta
\]

The term \(\frac{1}{\tau} \frac{\partial}{\partial x} (v \gamma)\) represents a local source of power from the shear strain. When integrated, this term accounts for the power generated at the beam ends. Besides this exchange with \(K_w\), the
shear energy density $\gamma_\gamma$ also directly trades local specific energy with the rotational kinetic energy $K_\omega$ through $\gamma_\Omega/\tau$. Finally, in addition to the latter exchange, $K_\omega$ trades energy with the potential-energy density $\gamma_\kappa$ from bending through the rotational local source

$$\frac{1}{\tau} \frac{\partial}{\partial x} (\Omega \kappa_b) = \Omega \frac{\partial \kappa_b}{\partial x} + \kappa_b \frac{\partial \Omega}{\partial x}$$

### 3.5 Results

#### 3.5.1 Parameter Range

As stated previously, the primary motivation for this work is to determine the force from debris impact. The ranges for the nondimensional parameters investigated are based on possible values for debris impact. A common U.S. design guideline (ASCE 2010) suggests that a prototypical debris object during flooding may be a 1,000 lb log. Wood properties vary considerably, but a nominal 9 m (29.5 ft) log with a diameter of 0.305 m (1 ft) are assumed here. For the column, the materials chosen are concrete, steel, and wood. For the concrete column, a square cross-section 0.6 m x 0.6 m is used (Mikhaylov 2009). For the steel column, a wide flange beam W24X104 is used; it was chosen so that it has nearly the same flexural rigidity ($EI$) as the concrete column. For the wood column, a diameter of 0.305 m is used, which is the same as the wood pole. The dimensional properties of the pole and columns are given in Table 1, and the corresponding nondimensional parameters are given in Table 2. For an isotropic material, 

$$\tau = 2(1+\nu)/\kappa$$

in which $\nu$ is the Poisson ratio, and a reasonable range of $\tau$ is from 20/9 at the low end to 5 or 6 at the high end. However, wood is not an isotropic material, and wood properties vary widely. A value of $\tau = 10$ is used here for the wood pole, which is at the low end of values suggested in Yoshihara et al. (1998).

The results presented here are nondimensionalized, but it is useful to obtain an understanding of the levels of force and duration that might be anticipated. An impact velocity of 10 km/h (6.2 mph) is physically reasonable during a tsunami. If the wood pole specified above were to hit a rigid stop axially, the impact force given by Eq.(3.12), based on $E = 9.3$ GPa (1,350 ksi) and $\tau = 10$, is 443 kN (99,600 lb). For a 9 m pole, the approximate duration is 4.22 ms. If it were to hit
a rigid stop transversely, the impact force given by Eq.(3.13) is 280 kN (62,950 lb) with an approximate duration of 6.7 ms.

For the columns, two lengths have been chosen: 3.6 m and 6 m. For the wood pole, two lengths have been chosen: 3 m and 9 m. For convenience in describing the results and for labels in the plots, the notation CiPi is defined such that C1 = 3.6 m column, C2 = 6 m column, P1 = 3 m pole, and P2 = 9 m pole. For example, a curve indicated by C1P2 shows the results of impact between a 3.6 m column and a 9 m pole. The length ratios $L_p/L$ for the different cases are shown in Table 3.

The results are organized as follows. First, the case of a pole hitting fixed-fixed and pinned-pinned columns will be presented. Second, results from Timoshenko and Euler beam theories are compared. This is followed by an examination of the effect of rotary inertia. Finally, the energy propagation and energy transfer is examined.

3.5.2 Axial Impact of a Pole against a Column

As mentioned previously, the spring between the column and the pole represents localized effects, which are difficult to quantify in practice and which will be effectively ignored herein. Packzkowski (2012) investigated the axial impact of a pole against a flexible, massless stop. They found that for ratios of spring stiffness to pole stiffness $k_r = k_s / (E_p A_p / L_p)$ of 40 or larger, the response was very nearly the same as for a rigid stop. To be conservative, herein $k_r = 100$ is used, and a parameter study has shown that the results presented are insensitive to higher values.

The impact force is nondimensionalized by Eq.(3.15). The time is nondimensionalized by either $t_{ax} = 2L_p / c_{ap}$, which is the time it takes for the impact axial wave in the pole to return back to the impact point, or $t_{sh} = L / c_s$, the time it takes for a shear wave in the column to return to the impact point. All impact occurs at the center of the column.
Table 1: Dimensional Properties

<table>
<thead>
<tr>
<th></th>
<th>$E$ (GPa)</th>
<th>$G$ (GPa)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood Pole</td>
<td>9.3</td>
<td>1.033</td>
<td>512.5</td>
<td>0.9</td>
</tr>
<tr>
<td>Wood Column</td>
<td>9.3</td>
<td>1.033</td>
<td>512.5</td>
<td>0.9</td>
</tr>
<tr>
<td>Concrete Column</td>
<td>24.682</td>
<td>10.284</td>
<td>2400</td>
<td>0.833</td>
</tr>
<tr>
<td>Steel Column</td>
<td>200</td>
<td>76.923</td>
<td>7800</td>
<td>0.393</td>
</tr>
</tbody>
</table>

Table 2: Nondimensional Parameters

<table>
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<tr>
<th></th>
<th>$\tau$</th>
<th>$\mu$</th>
<th>$\alpha_m$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole-Concrete column</td>
<td>2.88</td>
<td>1.328</td>
<td>0.043</td>
<td>1.049</td>
<td>21.472</td>
</tr>
<tr>
<td>Pole-Steel column</td>
<td>6.624</td>
<td>0.841</td>
<td>0.242</td>
<td>1.262</td>
<td>4.811</td>
</tr>
<tr>
<td>Pole-Wood column</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>2.581</td>
<td>1.632</td>
</tr>
</tbody>
</table>

Table 3: Length Ratios

<table>
<thead>
<tr>
<th>Case</th>
<th>$L_p / L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1P1</td>
<td>0.833</td>
</tr>
<tr>
<td>C1P2</td>
<td>2.5</td>
</tr>
<tr>
<td>C2P1</td>
<td>0.5</td>
</tr>
<tr>
<td>C2P2</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Figure 18 shows the impact force time history of the wood log hitting the fixed-fixed concrete columns. Time is nondimensionalized by $t_{ax}$. For C1P1 and C2P1, $t_{sh} / t_{ax}$ is 1.352 and 2.254, respectively. As a result, the impact force is dominated by the axial wave propagation, and separation occurs when $t \approx t_{ax}$. However, for C1P2 and C2P2, $t_{sh} / t_{ax}$ is 0.451 and 0.751, respectively. As a result, the returning shear wave in the column can be detected in the impact force at these times. Note that the time history of a pole hitting a rigid stop would be a
rectangular pulse of magnitude 1 and duration 1. C1P2 shows an increase above the initial impact force as a result of the vibration in the column.

Figure 19 shows the corresponding time histories for the steel columns. Time is again nondimensionalized by $t_{ax}$. For C1P1 and C2P1, $t_{sh}/t_{ax}$ is 1.299 and 2.165, respectively, and again the response is dominated by the axial wave. For C1P2 and C2P2, $t_{sh}/t_{ax}$ is 0.433 and 0.722, respectively. It is clear the returning shear wave results in a bigger disturbance in these cases, with a substantial amplification in the impact force from column vibration in the case of C1P2.

Considered together, Figure 18 and Figure 19 indicate that when a pole hits a stiff column, the duration of loading is determined by the time it takes the longitudinal wave to return to the impact point. However, if the shear wave in the column returns prior to this, the impact force can be increased substantially over the initial force.

The impact force for the pole hitting a wood column is shown in Figure 20. Time is nondimensionalized by $t_{sh}$. The sharp drop-off in the initial impact force evidenced in this figure is characteristic of a beam hitting a rigid stop. Hence, in all cases the impact force is dominated by the wave propagation in the column. However, if the longitudinal wave in the pole returns prior to the return of the shear wave, separation will occur. For C1P2, $t_{ax}/t_{sh}$ is 1.581, and the impact duration is determined by the shear wave propagation in the column, a result that is unattainable with Euler-Bernoulli beam theory (see section 3.5.3). Note also the sharp increase in impact force near the end of impact, which is a result of the dynamics of the column. For the other three cases, the impact duration is nearly equal to the time taken for the axial wave in the pole to return to the impact point, where $t_{ax}/t_{sh} = 0.527, 0.316, \text{ and } 0.949 \text{ for C1P1, C2P1, and C2P2 respectively.}$

Figure 21 compares the force-time histories of a wood pole hitting fixed-fixed and pinned-pinned steel columns for C2P2 (cf. Figure 19). Time is nondimensionalized by $t_{ax}$. The curves for both cases are identical until time $L/c_0 = 0.333 t_{ax}$. At one-half this time, the pinned support of the column shows rotation. That is, the wavefront in the column is propagating at $c_0$, ahead of
the shear wave propagating at approximately $c_t$. This wave causes rotation at the free end, then causes a slight change in impact force when the wave returns to the impact location. The wavefront propagating at $c_0$ is associated with the propagation of bending moment and angular velocity discontinuities (Boley and Chao 1955; Graff 1991). Note, however, that the impact force has a stronger dependence on the wave propagating at nearly $c_s$; these waves cause first a decrease and then an increase in the impact force. As long as the longitudinal wave in the pole has not returned prior to the waves in the column, the boundary conditions of the column can cause a significant difference in impact forces.

Figure 18: Impact force for a wood pole hitting fixed-fixed concrete columns
Figure 19: Impact force for a wood pole hitting fixed-fixed steel columns
Figure 20: Impact force for a wood pole hitting fixed-fixed wood columns
3.5.3 Comparison of Timoshenko versus Euler Models

The impact response from Timoshenko (T) and Euler (E) models for the column are compared in this section. A similar solution methodology as described above was implemented for the Euler model. Figure 22 shows the force-time histories for a wood pole hitting fixed-end concrete (C), steel (S) and wood (W) columns, for case C1P2. For the concrete column (circular symbols), both beam models give similar results. This is because the concrete column is quite stiff relative to the wood pole, and the response is dominated in both magnitude and duration by the axial response of the pole. The nearly constant impact force is characteristic of a pole hitting a stiff, massless stop. For the steel column (hollow diamond symbols), again it is quite stiff and the duration is also governed by the wave propagation in the pole. However, the reflected shear wave at time approximately 0.44 causes a drop in the impact force, and then the dynamics in the
column cause an increase. This behavior is captured poorly by Euler. For the wood column (solid diamond symbols), the force and duration are governed by the column. The initial peak followed by an exponential decay in the force is characteristic of a beam hitting a stiff, massless stop, for example. The Euler model substantially over predicts the initial peak force. In addition, the early return of waves in the column lead to poor prediction of the duration, which is governed by the shear wave in the column, as well as the intermediate response. These results indicate that Euler beam theory should not be used for these kinds of impact problems if the beam response is significant.

As observed in Figure 22 for the wood column, Euler can over predict the initial peak force, and as shown previously the magnitude is unbounded as the stiffness of the impact increases. Figure 23 shows the results for the wood pole hitting a wood column, C1P2, in which the contact stiffness has been varied. As noted previously, the stiffness ratio $k_r = 100$ has been used in all the previous results. Figure 23 also includes the results for $k_r = 50$ and 500. As can be seen, both Timoshenko and Euler give converged results for all cases except for the initial peak force as predicted by Euler. This force continues to increase as the spring stiffness increases, which is consistent with the derivation above that illustrated the potential unboundedness of the impact force for an Euler model. As noted previously, Euler also fails to predict well the time variation in the force.
Figure 22: Impact force for a wood pole hitting fixed-end concrete (C), steel (S), and wood (W) columns for case C1P2 based on Timoshenko (T) and Euler (E) beam theories.

Figure 23: Impact force for a wood pole hitting a fixed-end wood column for case C1P2 for different ratios of the contact stiffness to the axial pole stiffness, for Timoshenko (T) and Euler (E) theories.
3.5.4 Effect of Rotary Inertia

It is of interest to determine the influence of rotary inertia on the impact forces. Some structural finite element codes include shear deformation but ignore the rotary inertia. For this study, impact of a wood beam against a (nearly) rigid stop is considered. The beam has the same cross-sectional properties as before, with \( L/r = 120 \), \( \tau = 20/9 \), and a shear coefficient \( \kappa = 0.9 \). The rigid stop is modeled as a very stiff, massless spring, with a stiffness equal to 1200 times the shear stiffness of the beam.

The formulation cannot treat zero rotary inertia \( (\eta^2 = 0) \); therefore, this case was approximated by a beam with one-hundredth the normal rotary inertia \( (\eta^2 = 0.01) \). The results for impact with and without rotary inertia are shown in Figure 24. As can be seen, the rotary inertia in this case of nearly rigid impact for a beam with relatively high shear stiffness (small \( \tau \)) has a noticeable effect on the impact force near the end of impact, when the waves in the beam return to the impact point. The small oscillations for the case with rotary inertia that initiate at about time 0.67 result from the rotational kinetic energy propagating at \( c_0 \) returning to the impact point. With small (zero) rotary inertia, these waves have little (zero) energy. These results indicate that ignoring rotary inertia may not be sufficiently accurate if high fidelity in the impact force time history is desired.
Figure 24: Effect of rotary inertia on impact force for $L/r=120$ and $\tau = 20/9$ with $\eta^2 = 1$ and $\eta^2 = 0.01$

3.5.5 Energy Breakdown

The energy exchange gives an insight on the generation and behavior of the impact force. Figure 25 compares the total shear strain energy and the bending strain energy in the column as a function of time (nondimensionalized by $t_{sh}$) for the case of a wood pole hitting a fixed-fixed wood column (C1P2). The maximum impact force, shown in Figure 20, is dominated by shear, as can be seen by the sharp increase in shear strain energy and the much slower increase in bending strain energy.

Figure 26 shows the time histories for all strain and kinetic energies. The initial energy is all translational kinetic energy of the pole, but as the impact progresses a significant part of the initial pole translational kinetic energy is exchanged with the translational kinetic energy of the column, suggesting an important role for the inertia of the column on the impact dynamics. Also noteworthy is the initial energy exchange between the pole kinetic energy and shear and bending energy. The evolution of the bending energy requires the buildup of the curvature and a gradient of rotation (Eq.(3.21)). The shear energy on the other hand exchanges energy directly with the
kinetic energy of the pole (Eq. (3.20)). As a result, the initial exchange of the pole translational kinetic energy is mainly with the shear energy when compared with the bending energy, and the exchange is “local” in nature. As times progresses, curvature is established and pole kinetic energy is exchanged predominantly with translational kinetic energy of the column and bending energy of the column. Figure 27 shows the evolution of the various energy densities as functions of space and time during impact.

From Figure 27, it is apparent that at time of impact, there is a large and local spike in the shear energy density, which shows that the impact is absorbed initially via shearing. This local spike is then quickly transferred to bending energy as the spike in the shear energy density at the impact point propagates away from the impact point down the column until it is reflected at about time $L/2c_s$ ($\bar{t} = 1/2$); see Figure 27 (a). Although there is some wave dispersion after reflection, it is not as distinct as the dispersion of the flexural waves. Of particular interest, at the point of impact, $\bar{x} = 0$, the shear energy density reaches its maximum at the start of impact, and then drops to near zero. Away from the point of impact, the evolution of the rotational kinetic energy displays the richest dynamics. In fact, despite being of a smaller magnitude for this case, Figure 27 (d) shows that the disturbance in the rotational kinetic energy at the point of impact has two modes of wavefront propagation: at the distortional speed, $c_s$, and at the dilatational speed, $c_0$ (cf. Figure 21). The reflection of the waves for the rotational kinetic energy at the fixed ends is also more complex with several wavefronts being generated with pronounced lower frequency oscillations in their amplitudes. As predicted by theory (Boley and Chao 1955; Graff 1991), besides the rotational kinetic energy, bending energy also displays the second wavefront moving at dilatational speed, Figure 27 (b). Figure 27 (c) shows the evolution of the displacement kinetic energy of the column. It is apparent that a significant part of the initial energy is absorbed by the column at the impact point in the form of displacement kinetic energy. This energy propagates away from the impact point on the shear wave till the end of the column. At the end of the column, it is reflected and intensified by the strong shear strain gradient in the path of the shear wave. In addition, this figure shows that a simple mass spring model for the column would be inadequate, because the propagation of the kinetic energy by shear would be absent. Finally, Figure 27 (e) and Figure 27 (f) depict the passive role of the pole for this case. The pole serves as a mere signal propagator of the axial strain from the impact into the pole. In
particular, the wave motion on the pole has no effect on the duration of the impact, which is determined in this case by the return of the reflected shear wave from the column's ends.

Figure 25: Shear and bending strain energy in the wood column for case C1P2
Figure 26: Component energies and total energy for the case C1P2 with a wood column

(a) Shear energy of the column  (b) Bending energy of the column
3.6 Summary

An analytical solution for the linear impact response of a pole impacting a column, or a beam impacting a flexible axial stop, has been presented. The solution is based on Timoshenko beam theory and uses modal superposition. Contact can occur at an arbitrary point within the column span. The following conclusions may be made.

1. The initial peak impact force can be predicted by a simple formula, Eq. (3.15). In the case of stiff columns, the pole properties govern the force; in the case of a stiff pole, the column properties govern. In no case is the total mass of the projectile significant to the initial peak force.
2. The impact duration is governed by either the longitudinal wave speed in the pole (for stiff columns) or the shear wave speed in the column.

3. For cases in which the column properties are significant, the initial response is dominated by shear behavior, and Timoshenko beam theory rather than Euler-Bernoulli beam theory should be used. Indeed, based on Euler-Bernoulli beam theory the initial impact force is unbounded as the stop stiffness increases, whereas Timoshenko beam theory has a clear limiting value for the initial impact force.

4. The rotary inertia does not affect the initial peak impact force but it can have an effect on the impact force time history after waves in the beam return to the impact point.

5. The energy plots provide a vivid picture of the complex interactions and energy exchange taking place during the impact. For instance, of direct relevance to the present work, it explains the importance of the shear mode at the beginning of the impact to set the impact force and also depicts the relevance of the various energy exchanges in setting later spikes in the force as well as the impact duration. The energy exchange also shows the importance of the inertia of the column in absorbing a significant part of the initial kinetic energy of the pole.
Chapter 4

Transverse Impact of a Horizontal Beam on a Vertical Column

4.1 Introduction

Impact between a beam and a rod (i.e., a ‘stop’) or between two beams has various applications, including piping systems, valves, heat exchangers, and arrays of offshore oil production risers. However, the primary interest herein is the impact of water driven debris on coastal structures, for example when a wood pole or log impacts a structural column of a building or other structure.

Many of the studies on beam response during impact involve a mass impacting a beam rather than one beam impacting another beam transversely, which is the problem of interest here. In addition, many studies used Euler-Bernoulli beam theory, which gives inaccurate results for many transverse impact cases; see Khowitar et al. (2014). Moreover, there seems to have been little attention given to the energy exchange during multiple impacts of two beams.

In this chapter, Timoshenko beam theory is used to model the transverse impact of a pole on a column. Because our interest is primarily on the impact of water driven debris on coastal structures, which often involves small impact speeds (~3 m/s), only elastic behavior is considered. One important consequence of this assumption is that mechanical energy is conserved.

The pole is moving initially with a constant velocity and hits laterally the column at the midspan. A spring is placed between the pole and the column at the impact location. Although in principle such a spring could be used to represent a local, linear contact stiffness, in practice such stiffness is difficult to quantify. As in Khowitar et al. (2014), a very stiff spring is used such that the effect of the contact spring is negligible; it functions essentially as a penalty parameter to couple the pole and the column. Even a stiff contact spring as assumed here results in an impact force time-history that has a finite loading rate, which is more realistic than the discontinuous
time history that would result from an infinite contact stiffness. In addition, it eliminates the Gibbs phenomenon in impact simulations.

The primary focus in this chapter is on the contact force during multiple impacts and the energy exchange and behavior during each contact and separation phase. The chapter is organized as follows. First, the physical model is described and the governing equations and the nondimensionalization of the variables are presented. Then, the analytical solutions, which are based on superposition of normal modes of vibration, are developed. Results of a study on impact response for parameters within a range that might be seen in woody debris impact are presented and interpreted. Finally, the main findings of this study are summarized in the conclusions.

4.2 Physical System

Figure 28 shows a schematic of a pole hitting laterally a column. The pole moves towards the column with a constant translational velocity $v_0$ (at $x^*_1 = 0$) and angular velocity $\omega_0$ and hits anywhere along the pole. As mentioned, a contact spring with stiffness $k^*$ is placed between the pole and the column at the impact points. Multiple impacts are considered, and the whole collision event, from first contact to final separation, is divided into multiple contact phases and separation phases.

The column has fixed-fixed boundary conditions and zero initial conditions. The origin of the coordinate systems $x^*_1$ and $x^*_2$ ($x^*_1 = 0$ and $x^*_2 = 0$) are located at the point of impact. The column and the pole are considered elastic and homogeneous, with constant cross-sectional properties. The pole has Young’s modulus $E_p$, shear modulus $G_p$, mass density $\rho_p$, sectional area moment of inertia $I_p$, sectional area $A_p$, sectional rotary inertia $I_{pc} = \eta^2 p \rho_p I_p$, shear coefficient $\kappa_p$, and total length $L_p = L_{Ap} + L_{2p}$. A subscript “c” is used for the analogous column properties. The speeds of sound are $c_{0p} = \sqrt{E_p / \rho_p}$ and $c_{0c} = \sqrt{E_c / \rho_c}$. The speeds of shear waves are given by $c_{sp} = \sqrt{\kappa_p G_p / \rho_p}$ and $c_{sc} = \sqrt{\kappa_c G_c / \rho_c}$.
For this system, the fundamental kinematic variables, which are functions of time $t^*$, are the transverse displacement of the pole $w^*_p(x^*_p, t^*)$, cross-sectional rotation of the pole $\theta^*_p(x^*_p, t^*)$, transverse displacement of the column $w^*_c(x^*_c, t^*)$, and cross-sectional rotation of the column $\theta^*_c(x^*_c, t^*)$.

A similar nondimensionalization as in Boley (1955), Xing et al. (2002), and Khowitar et al. (2014) is used herein. The length scale is nondimensionalized by $r_p = \sqrt{I_p / A_p}$ and time by $r_p / c_{0p}$. The following nondimensional variables result:

$$\begin{align*}
x_1 &= \frac{x^*_1}{r_p}, \quad w_p = \frac{w^*_p}{r_p}, \quad x_2 = \frac{x^*_2}{r_p}, \quad w_c = \frac{w^*_c}{r_p}, \quad t = t^* \frac{c_{0p}}{r_p}, \quad k = \frac{r_p}{E_pA_p}
\end{align*}$$

The ratios of modulus of elasticity to effective shear modulus for the pole and column are $\tau_p = E_p / \kappa_p G_p$ and $\tau_c = E_c / \kappa_c G_c$. The ratio of the speed of sound in the two materials is $\mu_c = c_{0c} / c_{0p}$. $\mu_r = r_c / r_p$ is the ratio of the two radii of gyration, and $\mu_m = \rho_c A_c / \rho_p A_p$ is the ratio of the masses per unit length.
Figure 28: Schematic of pole and column

4.3 Equations of Motion

The equations of motion of the pole are

\[
\frac{1}{\tau_p} \left( \frac{\partial^2 w_p}{\partial x_1^2} - \frac{\partial \theta_p}{\partial x_1} \right) - k \left[ w_p(0,t) - w_c(0,t) \right] \delta - \frac{\partial^2 w_p}{\partial t^2} = 0
\]

(4.1)

\[
\frac{\partial^2 \theta_p}{\partial x_1^2} + \frac{1}{\tau_p} \left( \frac{\partial w_p}{\partial x_1} - \theta_p \right) - \eta_p \frac{\partial^2 \theta_p}{\partial t^2} = 0
\]

(4.2)

while the equations of motion of the column are

\[
\frac{\partial^2 w_c}{\partial x_2^2} - \frac{\partial \theta_c}{\partial x_2} - k \left[ w_p(0,t) - w_c(0,t) \right] \delta - \frac{\tau_c}{\mu_c} \frac{\partial^2 w_c}{\partial t^2} = 0
\]

(4.3)
\[
\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\mu_c^2 \tau_c} \left( \frac{\partial w}{\partial x} - \theta \right) - \frac{\eta_c^3}{\mu_c^2} \frac{\partial^2 \theta}{\partial t^2} = 0
\]  
(4.4)

in which \( \delta \) is the Dirac delta distribution. In the contact phase, there is a discontinuity in shear at \( x_1, x_2 = 0 \). Therefore the displacements are denoted as \( w_p^+ = w_p^+ \) for \( x_i \geq 0 \), \( w_p^- = w_p^- \) for \( x_i \leq 0 \), \( w_c^+ = w_c^+ \) for \( x_2 \geq 0 \), and \( w_c^- = w_c^- \) for \( x_2 \leq 0 \). Similar notation is used for \( \theta \).

The compatibility equations at \( x_i = 0 \) for the pole and at \( x_2 = 0 \) for the column are equal displacement, rotation, and curvature. In addition, we have the jump conditions for shear:

\[
\frac{\partial w_p^+(0,t)}{\partial x_i} - \frac{\partial w_p^-(0,t)}{\partial x_i} = \tau_p k \left[ w_p^+(0,t) - w_c^-(0,t) \right] = 0
\]

(4.5)

\[
\frac{\partial w_c^+(0,t)}{\partial x_2} - \frac{\partial w_c^-(0,t)}{\partial x_2} = \frac{\tau_c}{\mu_c^2 \mu_m} \left[ w_c^+(0,t) - w_c^-(0,t) \right] = 0
\]

(4.6)

### 4.4 Analytical Solution

#### 4.4.1 Separation of Variables

The method of separation of variables is adopted in which the kinematic variables can be written as

\[
\begin{bmatrix}
  w_p^+(x_1,t) \\
  \theta_p^+(x_1,t) \\
  w_c^+(x_2,t) \\
  \theta_c^+(x_2,t)
\end{bmatrix} = \sum_n q_n(t) \begin{bmatrix}
  \phi_p^+(x_1) \\
  \psi_p^+(x_1) \\
  \phi_c^+(x_2) \\
  \psi_c^+(x_2)
\end{bmatrix}
\]

(4.7)

The mode shapes of the pole and the column are the usual Timoshenko beam mode shapes. For the pole, the mode shapes can be written as (Khowitar et al. 2014)
\[
\begin{pmatrix}
\phi_p^+ \\
\psi_p^+ \\
\phi_c^+ \\
\psi_c^+
\end{pmatrix} = \begin{pmatrix}
A_1^+ \cos \beta_1 x_1 + A_2^+ \sin \beta_1 x_1 + A_3^+ e^{\alpha_1 t} + A_4^+ e^{-\alpha_1 t} \\
\kappa_{\beta_1} \left( A_1^+ \sin \beta_1 x_1 - A_2^+ \cos \beta_1 x_1 \right) + \kappa_{\alpha_1} \left( A_3^+ e^{\alpha_1 t} - A_4^+ e^{-\alpha_1 t} \right) \\
B_1^+ \cos \beta_2 x_2 + B_2^+ \sin \beta_2 x_2 + B_3^+ e^{\alpha_2 x_2} + B_4^+ e^{-\alpha_2 x_2} \\
\kappa_{\beta_2} \left( B_1^+ \sin \beta_2 x_2 - B_2^+ \cos \beta_2 x_2 \right) + \kappa_{\alpha_2} \left( B_3^+ e^{\alpha_2 t} - B_4^+ e^{-\alpha_2 t} \right)
\end{pmatrix}
\]

\[(4.8)\]

with

\[
\beta_1 = \sqrt{\frac{\omega^2 \left( \eta_p^2 + \tau_p \right) + \omega \sqrt{\omega^2 \left( \eta_p^2 - \tau_p \right)^2 + 4}}{\sqrt{2}}}
\]

\[
\alpha_1 = \sqrt{\frac{-\omega^2 \left( \eta_p^2 + \tau_p \right) + \omega \sqrt{\omega^2 \left( \eta_p^2 - \tau_p \right)^2 + 4}}{\sqrt{2}}}
\]

\[
\kappa_{\beta_1} = \left( \tau_p \omega^2 - \beta_1^2 \right) / \beta_1, \quad \kappa_{\alpha_1} = \left( \tau_p \omega^2 + \alpha_1^2 \right) / \alpha_1
\]

\[
\beta_2 = \sqrt{\frac{\omega^2 \left( \eta_c^2 + \tau_c \right) + \frac{\omega}{\mu_c} \sqrt{\mu_c^2 \omega^2 \left( \eta_c^2 - \tau_c \right)^2 + 4 \mu_c^2}}{\sqrt{2} \mu_c}}
\]

\[
\alpha_2 = \sqrt{\frac{-\omega^2 \left( \eta_c^2 + \tau_c \right) + \frac{\omega}{\mu_c} \sqrt{\mu_c^2 \omega^2 \left( \eta_c^2 - \tau_c \right)^2 + 4 \mu_c^2}}{\sqrt{2} \mu_c}}
\]

\[
\kappa_{\beta_2} = \left( \frac{\tau_c \omega^2 - \beta_2^2}{\mu_c} \right) / \beta_2, \quad \kappa_{\alpha_2} = \left( \frac{\tau_c \omega^2 + \alpha_2^2}{\mu_c^2} \right) / \alpha_2
\]

where \( \omega = \omega^* r_p / c_{0p} \) is the nondimensional natural frequency and \( \omega^* \) is the dimensional frequency. It should be noted that \( \beta_{1,2} \) are always real, but \( \alpha_{1,2} \) can be real or imaginary. The natural frequencies and the coefficients of the mode shapes are determined by imposing the compatibility and boundary conditions.

The generalized time function is
In transitioning between contact and separation phases, the ‘initial’ conditions at transition time $t_t$ must be transferred to the new phase. The general formulas for the coefficients $A_n$ and $B_n$ can be determined using the initial conditions and the property of orthogonality of the mode shapes from

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$$

where

$$A_n = I_w \cos \omega_n t - \frac{I_v}{\omega_n} \sin \omega_n t$$

$$B_n = I_w \sin \omega_n t + \frac{I_v}{\omega_n} \cos \omega_n t$$

$w_i$, $\theta_i$, $v_i$, $\omega_i$ are the transverse displacement, cross-section rotation, transverse velocity, and cross-section rotational velocity, respectively, at transition time $t_t$.

Two special mode shapes must be considered. The first are the rigid-body modes when $\omega_n = 0$, in which the mode shapes $(\phi, \psi)$ are $(x, 1)$ for rotational motion and $(1, 0)$ for translational motion, and the generalized time function is

$$q_n = A_n t + B_n$$

with $A_n = I_v$ and $B_n = I_w - I_v t_t$. The second is when $\omega \to \frac{1}{\eta_p \sqrt{\tau_p}}$, which means $\alpha_1 \to 0$, where the mode shapes can be obtained by taking the limit of Eq.(4.8).
4.4.2 Initial Peak Impact Force

For a very short time after the first impact, the deformation is dominated by shear, and the initial peak force can be derived the same way as in (Khowitar et al. 2014) for axial impact by satisfying the force equilibrium at the impact point. For a stiff contact spring, the initial peak force is

\[ F_0 = \frac{F_{pR} F_{cR}}{F_{pR} + F_{cR}} \]  

(4.12)

in which \( F_{pR} = 2\rho_p c_{sp} v_0 A_p \), \( F_{cR} = 2\rho_c c_{sc} v_0 A_c \), and \( c_{sp} \) and \( c_{sc} \) are the shear wave speeds in the pole and the column respectively. \( F_{pR} \) and \( F_{cR} \) are the initial impact forces for the pole and column, respectively, hitting a rigid stop; see Khowitar et al. (2014) for details.

It may be deduced from Eq. (4.12) that the initial peak force is dominated by local shear response and is independent of both pole and column lengths, and hence the total initial momentum and total mass. For design purposes, it is more convenient to write the impact force in terms of the shear stiffness of the pole, \( k_p = \kappa_p G_p A_p / L_p \), or of the column, \( k_c = \kappa_c G_c A_c / L_c \) and the respective total mass. Using the expressions for \( F_{pR} \) and \( F_{cR} \) in Eq. (4.12) results in

\[ F_0 = \mu_1 \sqrt{k_p \rho_p m_p v_0} \]  

(4.13)

with

\[ \mu_1 = \frac{2}{1 + \frac{1}{\mu_s \mu_n}} \]

or equivalently

\[ F_0 = \mu_2 \sqrt{k_c \rho_c m_c v_0} \]  

(4.14)

with
\[
\mu_z = \frac{2}{1 + \mu_c \mu_m}
\]

where \(\mu_c = c_m / c_p = \mu_c \sqrt{\tau_p / \tau_c}\).

From Eq.(4.13), it is clear that the peak impact force occurs when a (given) pole hits a rigid stop (i.e. \(\mu_c\) approaches infinity) and it is given by \(F_0 = 2\sqrt{k_p m_p v_0} = F_{pR}\).

### 4.4.3 Mechanical Energy

The energy exchange is useful to understand the behavior of the impact force during multiple impacts. For convenience, all energies are nondimensionalized by \(E_p A_p r_p\), and the nondimensional energies are

\[
TKE_p = \int_{-L_p}^{L_p} \frac{1}{2} \left( \frac{\partial w_p}{\partial t} \right)^2 dx
\]

\[
RKE_p = \int_{-L_p}^{L_p} \frac{\eta_p}{2} \left( \frac{\partial \theta_p}{\partial t} \right)^2 dx
\]

\[
BE_p = \int_{-L_p}^{L_p} \frac{1}{2} \left( \frac{\partial \theta_p}{\partial x} \right)^2 dx
\]

\[
SE_p = \int_{-L_p}^{L_p} \frac{1}{2} \left( \frac{\partial \omega_p}{\partial x} - \theta_p \right)^2 dx
\]

\[
TKE_c = \int_{-L_c}^{L_c} \frac{\mu_m}{2} \left( \frac{\partial w_c}{\partial t} \right)^2 dx
\]

\[
RKE_c = \int_{-L_c}^{L_c} \frac{\eta_c \mu_m}{2} \left( \frac{\partial \theta_c}{\partial t} \right)^2 dx
\]

\[
BE_c = \int_{-L_c}^{L_c} \frac{\mu_m \mu_c}{2} \left( \frac{\partial \theta_c}{\partial x} \right)^2 dx
\]

\[
SE_c = \int_{-L_c}^{L_c} \frac{\mu_m \mu_c}{2} \left( \frac{\partial \omega_c}{\partial x} - \theta_c \right)^2 dx
\]

\(TKE\) is the translational kinetic energy, \(RKE\) is the rotational kinetic energy, \(BE\) is the bending strain energy, \(SE\) is the shear strain energy, and “p” and “c” refer to pole and column, respectively.

To write the relationships between different energy components, it is convenient to represent the velocities as \(v_p = \partial w_p / \partial t\), \(v_c = \partial w_c / \partial t\), \(\Omega_p = \partial \theta_p / \partial t\), and \(\Omega_c = \partial \theta_c / \partial t\); the curvatures as
\( \zeta_p = \partial \theta_p / \partial x_1 \) and \( \zeta_c = \partial \theta_c / \partial x_2 \); and the shear strains as \( \gamma_p = \partial w_p / \partial x_1 - \theta_p \) and \( \gamma_c = \partial w_c / \partial x_2 - \theta_c \). The nondimensional equations of motion for the pole then become

\[
\frac{1}{\tau_p} \frac{\partial \gamma_p}{\partial t} - k \left[ \frac{\partial}{\partial x_1} \left( w_p(0,t) - w_c(0,t) \right) \right] \delta - \frac{\partial \gamma_p}{\partial t} = 0
\]  \hspace{1cm} (4.15)

\[
\frac{\partial \zeta_p}{\partial x_1} + \frac{1}{\tau_p} \gamma_p - \eta_p^2 \frac{\partial \Omega_p}{\partial t} = 0
\]  \hspace{1cm} (4.16)

and for the column

\[
\frac{\partial \gamma_c}{\partial t} - k \left[ \frac{\partial}{\partial x_2} \left( w_p(0,t) - w_c(0,t) \right) \right] \delta - \frac{\tau_c}{\mu_c} \frac{\partial \gamma_c}{\partial t} = 0
\]  \hspace{1cm} (4.17)

\[
\frac{\partial \zeta_c}{\partial x_2} + \frac{1}{\tau_c} \gamma_c - \eta_c^2 \frac{\partial \Omega_c}{\partial t} = 0
\]  \hspace{1cm} (4.18)

The nondimensional energy densities (per unit length) are as follows: \( \nu_{wp} = v_p^2 / 2 \) is the translational kinetic energy density of the pole, \( \nu_{wc} = \mu_m v_c^2 / 2 \) is the translational kinetic energy density of the column, \( \nu_{\theta p} = \eta_p^2 \Omega_p^2 / 2 \) is the rotational kinetic energy density of the pole, \( \nu_{\theta c} = \eta_c^2 \mu_m^2 \Omega_c^2 / 2 \) is the rotational kinetic energy density of the column, \( \nu_{\beta p} = \zeta_p^2 / 2 \) is the bending strain energy density of the pole, \( \nu_{\beta c} = \mu_m \mu_c^2 \zeta_c^2 / 2 \) is the bending strain energy density of the column, \( \nu_{\gamma p} = \gamma_p^2 / 2 \tau_p \) is the shear strain energy density of the pole, and \( \nu_{\gamma c} = \mu_m \mu_c^2 \gamma_c^2 / 2 \tau_c \) is the shear strain energy density of the column. Using Eqs. (4.15) and (4.16) one can obtain the governing equations for the pole in terms of the energy densities:

\[
\frac{\partial \nu_{wp}}{\partial t} = \frac{\nu_p}{\tau_p} \frac{\partial \nu_{\gamma p}}{\partial x_1} - k \left[ \frac{\partial}{\partial x_1} \left( w_p(0,t) - w_c(0,t) \right) \right] \delta
\]  \hspace{1cm} (4.19)
The governing equations for the column in terms of the energy densities can be obtained in a similar manner using Eqs. (4.17) and (4.18).

As discussed in Khowitar et al. (2014), these equations describe the interaction of the various energy densities and can be used to explain the energy transfer between the different energy components.

### 4.5 Solution Methodology

The analytical solution is obtained by following the same procedure used in Khowitar et al. (2014). The boundary and compatibility conditions can be written in terms of the mode shapes, which results in 16 equations to obtain the natural frequencies. A numerical method was used to obtain the natural frequencies of the system. The bisection method is used to narrow down the search domain and then the secant method was used to obtain the frequency. As discussed in the next section, six cases were considered here. A convergence study revealed that as a result of high harmonics during the impact, 500 symmetric mode shapes were sufficient for all cases. Some details regarding the solution are discussed in the appendix, as is a comparison with a finite element model.

### 4.6 Numerical Results

#### 4.6.1 Physical Dimensions

The results of a wood pole impacting a fixed-end column are considered here. The impact is at midspan of both the column and the pole, which is an important scenario for design. The pole has an initial translational velocity. The same material and dimensions are used for the pole and
the column as in the study conducted by (Khowitar et al. 2014). The pole has a weight of 342-kg (754-lb), length of 9-m (29.5-ft), and a diameter of 0.305 m (1 ft). This is somewhat less than the prototypical 1,000 lb wood pole suggested by a common U.S. design guideline (ASCE 2010). The materials of the columns are concrete, steel, and wood. The lengths of the columns are 3.6 m (C1) and 9 m (C2). For the concrete column, a 0.6 x 0.6 m cross-section is assumed. The steel column is a wide flange I-beam, W24X104, which has a bending stiffness $EI$ approximately the same as the concrete column. The wood column has the same diameter and material properties as the wood pole. These choices of columns vary from very flexible (long wood column) to very stiff (short concrete column). Therefore, the results reported herein should bound many useful practical cases.

Relevant dimensional properties and nondimensional parameters are shown in Table 4 and Table 5, respectively. A value of $\tau = 10$ for wood is used, which is at the low end of the values suggested in Yoshihara et al. (1998). A convergence study showed that the results do not change if the stiffness of the spring is 700 times the shear stiffness of the pole, $k_p$; therefore this was the minimum spring stiffness used.

The nondimensional time introduced previously is not physically very meaningful. Therefore, for presentation of the results time is nondimensionalized by the time the shear wave in the pole takes to return to the impact point, $\tilde{t} = t^* / t_p$, where $t_p = L_p / c_{sp} \approx 0.00668$ s.

Table 4: Dimensional Properties

<table>
<thead>
<tr>
<th>Member</th>
<th>$E$(GPa)</th>
<th>$G$(GPa)</th>
<th>$\rho$(kg/m$^3$)</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood pole and column</td>
<td>9.3</td>
<td>1.033</td>
<td>512.5</td>
<td>0.9</td>
</tr>
<tr>
<td>Concrete column</td>
<td>24.682</td>
<td>10.284</td>
<td>2,400</td>
<td>0.833</td>
</tr>
<tr>
<td>Steel column</td>
<td>200</td>
<td>76.923</td>
<td>7,800</td>
<td>0.393</td>
</tr>
</tbody>
</table>
Table 5: Nondimensional Parameters

<table>
<thead>
<tr>
<th>Case</th>
<th>τc</th>
<th>μm</th>
<th>μr</th>
<th>μc</th>
<th>μ1</th>
<th>μ2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole-concrete column</td>
<td>2.88</td>
<td>23.074</td>
<td>2.272</td>
<td>0.753</td>
<td>1.891</td>
<td>0.109</td>
</tr>
<tr>
<td>Pole-steel column</td>
<td>6.624</td>
<td>4.123</td>
<td>3.347</td>
<td>1.189</td>
<td>1.661</td>
<td>0.339</td>
</tr>
<tr>
<td>Pole-wood column</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The impact force is nondimensionalized by the instantaneous peak force obtained from Eq. (4.13). The dimensional instantaneous impact forces, per meter per second, are 98 kN, 86.5 kN, and 50.4 kN for the concrete, steel, and wood columns, respectively. It can be seen from Table 5 and Eq. (4.13) that the variation in these values depends mainly on the mass ratio, which ranges between 1 and 23 and is the parameter that has the most variation for these cases.

4.6.2 Force-Time History

Figure 29 shows the impact force-time history for a 9 m wood pole hitting 3.6 m and 6m concrete, steel, and wood columns. Note that in all cases the initial peak force is predicted well by Eq. (4.13), i.e., the initial nondimensional peak force is nearly 1. However, the maximum impact force is up to 1.5 times the initial peak.

For the concrete columns, the duration of the first contact phase is 0.00668 s, which is the time taken by the shear wave in the pole to return to the impact point. At that time, the velocity along the pole away from the impact point is still for the most part the initial velocity. The forward momentum leads the portion of the pole at the impact point to move and hit the column again after a short time. For most of the impact cases, there is a high frequency pounding after a fairly large separation phase. However, this is not the case for the most flexible (long wood) column, in which the durations of all the contact and separation phases are small. When the shear wave in the column returns to the impact point prior to separation, the force experiences a sudden change and subsequent high harmonics during the first contact phase. For instance, the shear wave in the column returns to the impact point at $\tilde{t} = 0.29$ for the short concrete column (Figure 29a), 0.48 for the long concrete column (Figure 29b), and 0.27 for the short steel column (Figure 29c).
It can be noticed from Figure 29 that for concrete and steel columns, the nondimensional duration of the impact event is between 5.5 and 6. However, for the wood column, the durations are 7.1 and 10.4. This is because the more flexible wood columns displace more, and the pole and column move more together and final separation takes longer. The higher frequency of the contact events results from the smaller relative displacement between the pole and the column for these cases.

(a) 3.6 m concrete column  
(b) 6 m concrete column  
(c) 3.6 m steel column  
(d) 6 m steel column
4.6.3 Energy-Time History

Figure 30 shows the time histories of energy for the six cases. The energy is nondimensionalized by the initial kinetic energy of the pole. The rotational kinetic energies and the spring energy are omitted from the plots because their values are very small relative to the other energies.

For all cases, most of the energy is in bending strain and translational kinetic energies of the pole. However, it can be seen that in a very short time after the first impact, the shear energy dominates the bending energy, which confirms that the instantaneous impact force is dominated by shear deformation either in the pole or in the column. The bending energy then dominates as the effect of the impact propagates along the pole and column. Since the stiffness of the column is relatively high for the case of the concrete and steel columns, the sum of the bending strain energy and the kinetic translational energy in the pole constitutes almost 90% of the total energy during the collision event. Moreover, these two energies are almost symmetric around a horizontal line at 45% of the total energy. On the other hand, for the soft (wood) column, a significant part of the total energy is absorbed by the bending strain energy and translational kinetic energy of the column.
For the concrete and steel columns, the kinetic translational energy of the pole decreases with the increase of the bending strain energy until \( \tilde{t} \approx 0.8 \), at which time the impact force experiences a sharp increase, the kinetic translational energy starts to increase and the bending strain energy starts to decrease. At \( \tilde{t} \approx 1.35 \) the translational kinetic energy of the pole reaches a relative maximum and the bending energy reaches a relative minimum value, at which the second contact phase almost starts. After that the kinetic energy decreases while the bending energy increases. When \( \tilde{t} \approx 3 \), the bending strain energy of the pole reaches a relative maximum value and the kinetic translational energy reaches a relative minimum value, and then the bending energy gradually decreases with the increase of the kinetic energy until a local drop at \( \tilde{t} \approx 4 \) occurs in the bending energy at which the separation phase with the larger duration almost ends. After the bending energy reaches a local minimum at \( \tilde{t} \approx 4.2 \), the bending energy of the pole then starts to increase with the decrease of the kinetic translational energy until it reaches another peak value at \( \tilde{t} \approx 4.8 \) at which the translational kinetic energy has another minimum value. This basically antisymmetric energy exchange between these two components is a reflection of the very low energy in the other components of the pole and the stiffness of the columns. It can be seen that the bending strain energy and the translational energy of the pole are equal at \( \tilde{t} \approx 2 \) and \( \tilde{t} \approx 5.5 \).

On the other hand, for the short wood column, the kinetic translational energy reaches its minimum value at \( \tilde{t} \approx 3.5 \), at which the bending energy of the pole reaches its peak value. They are equal at \( \tilde{t} \approx 2.4 \) and \( \tilde{t} \approx 6 \). The bending energy of the column has peaks at \( \tilde{t} \approx 1 \) and \( \tilde{t} \approx 6 \). For the case of the long wood column, the kinetic translational energy of the pole attains its minimum value at \( \tilde{t} \approx 5 \), at which point the bending strain energy of the pole attains its peak value. They are equal at \( \tilde{t} \approx 2.9 \) and \( \tilde{t} \approx 6.8 \). The bending energy of the column has peak values at \( \tilde{t} \approx 2 \) and \( \tilde{t} \approx 8.5 \).

It is interesting to note how much energy is transferred to the column at the final separation. For the most rigid (short concrete) column, only 1.5% of the initial energy is transferred to the column. For the most flexible (long wood) column, almost 7.2% of the initial energy is transferred to the column. In this case, at the time of separation most of the energy transferred to the column is absorbed by bending strain energy. In all cases there is relatively little net energy.
transfer to the column. This means that the impact will have a similarly small impact on the structure of which the column is a part. However, it should be noted that the fixed end conditions of the columns likely reduce the net energy transfer.

(a) 3.6 m concrete column                      (b) 6 m concrete column

(c) 3.6 m steel column                          (d) 6 m steel column
Figure 30: Component energies and total energy

4.6.4 Impulse-Time History

Figure 31 shows the impulse time histories for the six cases. The impulse is nondimensionalized by the initial momentum of the pole. The plot shows that the final impulses for the concrete and steel columns are almost 1.6 times the initial momentum of the pole. For the wood column, the final impulse is almost 1.8 times the initial momentum of the pole. This indicates that the final mass-averaged rebound velocity of the pole is almost 60% of the initial velocity for the concrete and steel columns, and 80% of the initial velocity for the wood column. These results are consistent with the energy plots in Figure 30.
(a) 3.6 m concrete column
(b) 6 m concrete column
(c) 3.6 m steel column
(d) 6 m steel column
Figure 31: Impulse time histories a wood pole hitting columns of different lengths and materials

4.6.5 Energy Decomposition by Modes

The previous energy plots depict the evolution of the extensive energy quantities. They show the bulk exchange between the energy components, but are blind to the local processes of this exchange. This and the next section discuss these local processes.

Figure 32 shows the mode contribution of bending and shear energies of the pole for the short concrete column (left) and long wood column (right). The nondimensional times are 5.5 and 10.2 for the short concrete column and the long wood column, respectively, which correspond to the durations of the collision event.

These figures clearly show the sharp distinction between the shear and flexural contributions to the beam vibrational motion. The evolution of the flexural energy, shown in Figure 32(a), is dominated by the small wave number modes, and associated low period motion. These modes aggregate most of the flexural energy regardless of the relative stiffness between pole and column. In addition, their characteristic time scale is much larger than each individual contact phase, which results in a time evolution that is less affected by the impact events.
By contrast, the shear modes, shown in Figure 32(b), are characterized by short period vibration and a broader energy spectrum. The increased energy contained in the higher wave number modes reveal the local nature of the impact, and shows the dominance of the shear at the early instants of the impact. Indeed, as shown in (Khowitar et al. 2014), it is the shear stress that sets the magnitude of the peak force. The prevalence and locality of the shear forces at the initial instants of the impact are also the fundamental assumptions underpinning Eq. (4.12). This equation shows good agreement between the initial peak-forces obtained with the exact solution of the Timoshenko beam equations and Eq. (4.12). And, moreover, Eq. (4.12) shows that the incoming total momentum is immaterial to the computation of the peak force.

Lastly, but not less important, the modal analysis in Figure 32 shows a broadening of the spectrum of the shear energy over time. The broadening is more pronounced for the impact of the pole against the more compliant long wood-column and results from the enhanced energy exchange and increased impact phases between the pole and column in this case.
Figure 32: Modal contribution of energy of the pole for short concrete column (left) and long wood column (right)

4.6.6 3D Energy Density:

Figure 33 shows the energy density for the short concrete column (on the left) and the long wood column (on the right). These figures complement the modal analysis of the previous section with spatiotemporal plots of the energy density for each of the participating energy components in the pole. A salient feature of these plots is the irregular nature of the vibrational motion after a series of impact events.

As shown in Figure 33, the two dominant energy components for the full extent of the impact are the translational kinetic energy and the bending energy of the pole. The bulk of the former and latter components travel at the shear speed $c_s$, but in the case of bending a small portion of the energy also travels at dilatational speed $c_0$. In the observed vibrational motion, the bulk motion of the translational kinetic energy represents the large modes that retain a degree of organization over an appreciable extent of time, especially for the impact against the concrete column. These large structures can still be seen at the initial stages of impact on the long wood, but close to the end of all impacts seem to broadly mix along the half beam. The impact with the long wood also displays an accumulation of kinetic energy at the free end of the beam that shows a spatial complementarity with flexural energy, in addition to the temporal complementarity shown in Figure 33 by the corresponding extensive quantities. Thus bending accumulates at the impact point, where the beam curvature is largest. Concurrently with these regular mode
propagations, dispersion, reflection and interaction are also of import to the impact dynamics. The latter is more frequent for the impact with the long wood, and together with dispersion and reflection yields extraordinarily small time scale motions for the rotational kinetic energy and potential shear energy at the pole. Rotational kinetic energy density is null at the impact point for the impact at mid beam and grows towards the free end of the pole, where it displays a “whipping” effect. The shear energy is fed from the column at the impact point and quickly spreads to each half beam and becomes choppy throughout the beam. In fact, Figure 33(b) shows an unexpectedly high amplitude low mode for shear energy in the impact against the long wood that results from the nearly continuous mutual pounding between pole and column.

(a) Translational kinetic energy density
(b) Rotational kinetic energy density

(c) Bending energy density
Figure 33: Time variation of energy densities of the pole for short concrete column (left) and long wood column (right)

4.7 Computational Aspects

The characteristic equation of the natural frequency $\omega$ was solved numerically to obtain the roots using Mathematica 9.0. A frequency increment of 1/200,000, starting at zero, was used to narrow down the search domain around the root, and then the secant method was used to obtain the roots. Due to the high harmonics during the impact, high precision (350 decimal digits) was used to obtain the roots and the response. For all cases, 500 symmetric mode shapes were used to obtain the response. To decrease the analysis time, symmetric geometry was also utilized, i.e., integrations were carried out only over half the pole and column. In a desktop computer (3 GHz Intel Core i5, with four cores and 12 GB RAM memory, running Windows 8.1), the time taken for each phase is approximately 45 minutes, resulting in total times of the collision event between 720 minutes (for 6 m steel column with 16 phases) to 6570 minutes (for 6 m wood column with 146 phases).

In addition to the analytical solution described above, a finite element solution was also utilized. The analytical solution was used to verify the finite element solution. For the finite
element model, both the pole and the column are modeled using a standard shear-deformable beam element that uses the exact static stiffness. A lumped mass matrix was used. It has been shown that the time history of the impact force can be influenced by the rotary inertia (Khowitar et al. 2014), so lumped rotational inertia was included in the mass matrix.

Several runs were performed to assess the correctness of the FEM solution, as well as to determine the mesh and time step requirements for accurate simulations. The converged mesh consisted of 4000 elements for the pole and 3000 elements for the longest column. The Newmark method was used for a direct integration of the equations of motion. For all cases, a nondimensional time step of $1.5 \times 10^{-4}$ was used. Figure 34 compares the analytical and finite element force-time histories for a 6-m wood pole hitting a 3.6 m concrete column. The results obtained from the two approaches match very well.
4.8 Summary

The linear response of a beam hitting transversely a fixed-fixed column is studied in this paper. The results presented are for a 9 m wood pole hitting transversely concrete, steel, and wood columns having lengths of 3.6 m and 6 m. The pole moves with an initial velocity and hits laterally the column at midspan. Multiple impacts were considered and the whole collision event is divided into contact and separation phases. Results from the analytical solution were used to verify the finite element solution. In all cases excellent matches were obtained. The following conclusions may be made.
1. The initial peak force is well-predicted by Eq. (4.13). However, the maximum impact force can reach up to 1.5 times the initial peak force.

2. Variation in initial peak impact force depends mainly on the mass ratio of the column and the pole.

3. The instantaneous impact force is dominated by shear deformation either in the pole or in the column, or both.

4. For most of the cases, there is a high frequency pounding after a fairly large separation phase. However, for the most flexible (long wood) column, the durations of all contact and separation phases are small due to the high frequency pounding between the pole and the column.

5. For concrete and steel columns, the sum of the bending strain energy and the kinetic translational energy in the pole constitutes almost 90% of the total energy during the collision event.

6. For the soft (wood) column, a significant part of the total energy is absorbed by the bending strain energy and translational kinetic energy of the column during the collision event.

7. For the most rigid (short concrete) column, only 1.5% of the initial energy is transferred to the column. However, for the most flexible (long wood) column, almost 7.2% of the initial energy is transferred to the column. In this case, at the time of separation most of the energy transferred to the column is absorbed by bending strain energy at final separation.

8. The total impulses for the concrete and steel columns are almost 1.6 times the initial momentum of the pole. For the wood columns, the total impulse is almost 1.8 times the initial momentum of the pole. This indicates that the final mass-averaged rebound velocity of the pole is almost 60% of the initial velocity for the concrete and steel columns, and 80% of the initial velocity for the wood columns.
Chapter 5
Additional Scenarios

5.1 Introduction

In this chapter, additional scenarios for axial and transverse impacts that supplement the results in the previous two chapters are considered. The axial impact simulations presented in Chapter 3 only considered the first contact phase. In this chapter, the same scenarios are simulated but through the entire impact event, capturing the multiple impacts, if any. The maximum impact force, which normally does not occur during the first contact phases, is sought. Moreover, the main differences between cases that exhibit single and multiple impacts will be clarified from the energy exchange. For transverse impact, free-end and pin-end column boundary conditions are considered to supplement the fixed-end boundary conditions in Chapter 4. The free-end condition, while not particularly relevant for a pole hitting a column, has some interest in the transverse impact of two poles, for example.

In attempt to assess the values of the initial peak force obtained herein for design purposes, it is useful to give an insight on whether applying this force as a static load can bound the dynamic response. The values of the maximum shear force and bending moment are obtained by applying a static load of a value equal to the initial peak force at mid-span of a fixed-end column. These values are then compared to the values of the shear force and the bending moments from the dynamic analyses.

As compared to the analytical results in the previous chapters, the results in this chapter were obtained using a finite element program. As shown in Section 4.7, if modeled appropriately the finite element solutions agree very well with the analytical solutions. For the long pole, 4000 elements were used and 3000 elements were used for the long column. The Newmark method was used for direct integration of the equations of motion. For all cases, a time step of $10^{-6}$ s was used.
5.2 Axial Impact

Axial impact will be discussed for a wood pole hitting concrete, steel, and wood columns. For each column material there will be four cases C1P1, C1P2, C2P1, and C2P2. C1 indicates a column of length 3.6 m, C2 indicates a column of length 6 m, P1 indicates a pole of 3 m, and P2 indicates a pole of 9 m. These are the same cases as considered in Chapter 3.

5.2.1 Wood pole hitting concrete column

For the axial impact of a pole hitting a concrete column, the collision event is characterized by a single impact. The reason for this is that the high stiffness of the column relative to the pole leads to very small displacements of the column at the impact point. Therefore, the force-time plots are not repeated here because they are identical to those presented in Chapter 3, Figure 18.

Figure 35 shows the time evolution of the energy components of the pole and the column. The time is nondimensionalized with respect to $t_{ax} = \frac{2L_p}{c_{op}}$, which is the time it takes for the axial wave in the pole to return back to the impact point. The energy is nondimensionalized with respect to the initial kinetic energy of the pole. The symbols in the plots are: TE; total energy, TKEp; translational kinetic energy of the pole, ASEp; axial strain energy of the pole, TKEc; translational kinetic energy of the column, RKEc; rotational kinetic energy of the column, BEc; bending energy of the column, SEc; shear energy of the column, and SprE; strain energy of the spring. All impacts occur at the center of the column.

It can be seen from Figure 35 that most of the kinetic energy of the pole is transferred into axial strain energy in the pole. This is due to the relatively high stiffness of the column, which leads to higher deformation in the pole relative to the column. It is noticed also that the bending energy of the column is smaller than the translational kinetic energy of the column for all cases except for the case C1P2, in which the shear wave in the column reaches the impact point at 0.45 $t_{ax}$ and causes the impact force to increase. As stated previously, shear energy of the column is much larger than the bending energy of the column for a very short time after the first impact, which indicates that the initial peak impact force is dominated by the local shear deformation of the column at the impact point. In addition, kinetic and axial energy of the pole reaches their minimum and maximum values, respectively, when the axial wave in the pole reaches its far end.
Figure 35: Component energies and total energy for a wood pole hitting axially a concrete column

Figure 36 shows the impulse-time history. The impulse is nondimensionalized with respect to the initial momentum of the pole. It can be seen that the total impulse at the final separation is approximately 1.8 times the initial momentum of the pole. This implies the mass-averaged rebound velocity of the pole is around 80% of the initial velocity of the pole.
As discussed in the previous section, most of the energy is transferred into axial strain energy of the pole (almost 80%), and since the impact is elastic, the pole returns to its original shape after separation occurs, releasing the strain energy in the form of kinetic energy. Thus, the rebound velocity of the pole is around 80% of the initial velocity of the pole.

Figure 36: Impulse for a wood pole hitting axially a concrete column
5.2.2 Wood pole hitting steel column

For the scenario of a wood pole hitting a steel column, the only case that exhibits multiple impacts is C2P2. In this case, the flexibility of the steel column is sufficient to allow the column to vibrate with larger amplitudes. Also, the long pole has a bigger mass, increasing the inertia of the pole. The force-time plots for the cases of C1P1, C1P2, and C2P1 are shown in Chapter 3, Figure 19.

Figure 37 shows the force-time history of the case C2P2. The force is nondimensionalized with respect to the maximum instantaneous force obtained from Eq. (3.15) and the time is nondimensionalized with respect to \( t_{ax} = 2L_p / c_{wp} \). It can be seen that in the first contact phase, there is a sharp decrease and then increase in the impact force at the time when the shear wave in the column returns to the impact point. The plot shows also that the maximum impact force occurs in the second contact phase and it is approximately 1.6 times the initial peak force.

![Figure 37: Force-time history for case C2P2 of a wood pole hitting axially a steel column](image)

Figure 37 shows the energy components versus time. It can be seen that for the cases C1P1 and C2P1, the pole is short and separation occurs before the shear wave in the column returns to the impact point. For these two cases, the impact does not cause much bending deformation in
the column due to the relatively small mass of the pole. Hence, the translational kinetic energy of the column is larger than the bending energy of the column throughout the entire collision event. It is interesting also to note that the shear energy for these two cases exceeds the bending energy, which shows that most of the deformation in the column is due to shear rather than bending. However, for the case of C1P2, the bending energy of the column exceeds the translational kinetic energy of the column. Also, when the shear wave in the column returns to the impact point, the impact force experiences a sharp increase. Moreover, the shear energy of the column is always larger than the bending energy during the collision event. For the first three cases, there is only a single impact and most of the initial kinetic energy of the pole is transferred into strain energy of the pole. For the last case C2P2, at the beginning of the impact, the axial strain energy of the pole absorbs most of the kinetic energy of the pole, and reaches its peak value at $\bar{t} = 0.5$. It can be seen that the bending energy of the column has a small value, but it grows gradually until it reaches its peak value at $\bar{t} \approx 1$. The shear energy exceeds the bending energy shortly after the first impact, which emphasizes the dependence of the initial peak force on the local shear deformation of the column. The translational kinetic energy of the column attains its maximum value at the beginning of the second contact phase, and then decays rapidly until the final separation.

(a) C1P1

(b) C1P2
Figure 38: Energy components and total energy for a wood pole hitting axially a steel column.

Figure 39 shows the time evolution of the impulse. It can be seen that for cases C1P1 and C2P1, the final impulse is approximately 1.5 times the initial momentum of the pole. However, for the case C1P2, the final impulse is around 1.8 times the initial momentum of the pole. In this case, the slope of the curve increases when the shear wave returns to the impact point and the force starts to increase sharply. Figure 39 (d) for the case C2P2 shows that the final impulse is approximately 1.7 times the initial momentum of the pole, which indicates that the mass-averaged rebound velocity of the pole is approximately 70% of the initial velocity.
5.2.3 Wood pole hitting wood column

All cases of axial impact of a pole hitting a wood column involve multiple impacts due to the relatively small stiffness of the column. Figure 40 shows the force-time histories. The time is nondimensionalized with respect to \( t_{sh} = L / c_s \), the time it takes for a shear wave in the column to
return to the impact point. The results show that the maximum impact force always occurs near the end of the collision event and can reach up to 1.9 times the initial peak force. It can be noticed that for the case of a short pole, C1P1, and C2P1, the duration of the entire collision event is smaller than that for the long pole cases C1P2 and C2P2, which indicates that the time of the final separation depends on the wave propagation in the pole.

![Graphs showing impact force over nondimensional time for different cases](image)

(a) C1P1  
(b) C1P2  
(c) C2P1  
(d) C2P2

Figure 40: Impact force for a wood pole hitting axially a wood column
Figure 41 shows the time evolution of the energy components. It can be seen that the axial strain energy of the pole is very small, and the translational kinetic energy of the column absorbs most of the kinetic energy of the pole at the beginning of the impact. This indicates that the displacement of the column at the impact point prevents much of the axial deformation of the pole as they move together in the same direction. The shear energy of the pole at the very beginning of the impact exceeds the bending since the deformation in the column is mostly due to shear. The bending energy then builds up gradually and attains its local peak value approximately around half of the final separation time.
Figure 41: Energy components and total energy for a wood pole hitting axially a wood column.

Figure 42 shows the time evolution of the impulse. The final impulse ranges between 1.7-1.9 times the initial momentum of the pole, and thus the mass-averaged rebound velocity of the pole is between 70% and 90% of the initial velocity.
Figure 42: Impulse for a wood pole hitting axially a wood column

5.3 Transverse Impact

Multiple impacts are investigated for a 9 m wood pole hitting transversely a 6 m wood column having free and pinned end boundary conditions. The time is nondimensionalized with respect to \( t_{sh} = L_p / c_{yp} \), the time it takes for a shear wave in the pole to return to the impact point.

Figure 43 shows the force-time histories of a wood pole hitting a free-end wood column (left) and pinned-end wood column (right). The force is nondimensionalized with respect to the initial peak force obtained from Eq. (4.13). It can be seen from Figure 43 (a) that the maximum force during the entire collision event almost equals the peak instantaneous force in the first contact phase. This is because the free-end column vibrates with lower frequencies than the end-restrained column. The entire column moves in the direction of the initial velocity while vibrating with smaller amplitudes after each impact. The figure shows also a separation phase with a long duration from \( T \approx 3.4 \) to \( T \approx 8 \) where the column moves with a constant velocity. Of all the cases for different end conditions of the column, the free-end boundary condition gives the shortest duration of the entire collision event.
Figure 43 (b) shows the case of a pinned-end column. The maximum force is almost 1.48 times the initial peak force. The free rotations at the ends of the column allow the column to vibrate with higher amplitudes, and hence the duration of the entire collision event is longer than that for the fixed-end column. The maximum force has a smaller value than that for the fixed-end column because of the lower frequencies and the bigger displacements that the pinned-end column undergoes at the impact point.

![Graph showing impact force vs nondimensional time for free-end and pinned-end columns.](image)

(a) free-end column  
(b) pinned-end column

Figure 43: Impact force for 9 m wood pole hitting transversely a 6 m wood column

Figure 44 shows the time evolution of the energy components for the free-end column (left) and for pinned-end column (right). It can be seen from Figure 44 (a) that most of the kinetic energy of the pole is transferred into translational kinetic energy of the column, which exceeds the bending energy of the column throughout the whole collision event. It is noticed that between $\tilde{t} \approx 3.4$ and $\tilde{t} \approx 8$ the column moves with constant velocity in the same direction of the initial velocity. The bending energy of the column exceeds the bending energy of the pole for a short time after the first impact, and then the bending energy of the pole exceeds the bending energy of the column and reaches its peak value at half of the duration of the collision event. The shear energy of the column is very small compared to the other components and the bending energy becomes very small starting from the fifth contact phase at $\tilde{t} \approx 2.7$. 
Figure 44 (b) shows that the translational kinetic energy of the column exceeds other energy components for a short time after the initial impact and reaches its maximum value at $t \approx 1$. It is observed that most of the kinetic energy of the pole is transferred into bending energy of the column. The shear energy of the column exceeds the bending energy of the column for a very short time after the initial impact, then the bending energy builds up until it reaches its maximum value almost at half of the impact duration.

![Energy components and total energy](image)

(a) free-end column     (b) pinned-end column

Figure 44: Energy components and total energy for a 9 m wood pole hitting transversely a 6 m wood column

Figure 45 shows the impulse-time history for a free-end column (left) and pinned-end column (right). The impulse is nondimensionalized with respect to the initial momentum of the pole. It can be noticed from Figure 45 (a) that the final impulse is only approximately 0.6 times the initial momentum of the pole. This indicates that the final mass-averaged velocity of the pole is almost 60% of the initial velocity. It also indicates that the pole and the column are moving in the same direction of the initial velocity throughout the entire collision event.

Figure 45 (b) shows that for the pinned-end column, the final impulse is almost 1.9 times the initial momentum of the pole. This indicates that the mass-averaged rebound velocity is almost 90% of the initial velocity.
Maximum Shear and Bending Envelopes

In attempt to assess the values of the initial peak force obtained herein for design purposes, it is useful to give an insight on whether applying this force as a static load bounds the dynamic response. The values of the maximum shear force and bending moment are obtained by applying a static load of a value equal to the initial peak force at mid-span of a fixed-end column. These values are then compared to the values of the shear force and the bending moments from the dynamic analyses. The initial peak force for the axial impact, $F_0$, is obtained from Eq. (3.15) and for transverse impact from Eq. (4.13). The dynamic amplification factors are $R_s = SF_D / SF_S$ and $R_b = BM_D / BM_S$, where $SF_D$ is the maximum dynamic shear force, $BM_D$ is the maximum dynamic bending moment, $SF_S = F_0 / 2$ is the maximum static shear force, $BM_S = F_0 L_c / 8$ is the maximum static bending moment, and $L_c$ is the length of the column.
5.4.1 Envelope versus time

5.4.1.1 Axial Impact

Figure 46 shows the maximum shear and bending moment envelopes with respect to the nondimensional time for axial impact. The nondimensionalization of the time depends on the impact case where \( t_{at} \), \( t_{sh} \), and \( C_i P_i \) were defined previously.

It can be seen that the case C1P2 gives the maximum shear amplification factor for single impact, and C2P2 for multiple impacts. For all cases, case C1P2 always gives the maximum bending moment amplification factor. This is due to the high frequency vibrations of the column that depend on the stiffness of the column and inertia of the pole. For a single impact, case C1P2 has the maximum bending and shear force amplification factors. For steel column, C1P2 has the maximum moment, and C2P2 has the maximum shear due to the multiple impacts. For wood column, C2P2 gives the maximum shear force, and C1P2 gives the maximum bending moment. For all column materials, the dynamic shear force equals the static shear force shortly after the first impact. For concrete and steel columns, there is a jump in the maximum shear force when the shear wave returns to the impact point. For all cases, the maximum shear force amplification factor is 3.87 (refer to Table 6 for maximum dynamic amplification factors) and it is attained for the case of steel column C2P2.

(a) Concrete column
Figure 46: Maximum shear force (left) and bending moment (right) amplification factors for axial impact

5.4.1.2 Transverse Impact

It can be seen from Figure 47 that short concrete and steel columns give the largest shear force amplification factors. Largest dynamic shear force always occurs at later impacts. It can be noticed that in all cases the maximum dynamic shear force is equal to the maximum static shear
force shortly after the first impact. The maximum values of the shear force amplification factor are 4.71 and 4.15 for short concrete and steel columns, respectively. The short concrete and steel columns also give the maximum bending moment amplification factor with values greater than 1. However, for all wood column cases as well as the long concrete and steel columns, the values of the bending moment amplification factor are less than 1.

(a) 3.6 m concrete column

(b) 6 m concrete column
(c) 3.6 m steel column

(d) 6 m steel column
Figure 47: Maximum shear force (left) and bending moment (right) amplification factors for transverse impact
5.4.2 Envelope versus space

5.4.2.1 Axial Impact

It can be seen from Figure 48 that for a concrete column, the maximum dynamic shear force equals the maximum static shear force at the impact point. Case C1P2 gives the maximum shear force amplification factor at the fixed supports. For a steel column, C1P1 and C2P1 (short pole) give \( R_s = 1 \) at the impact point. The shear force amplification factor for the steel column for case C1P2 is greater than that for C2P2 up to 1/3 of the column length measured from the impact point, after that it increases for case C2P2 and reaches its maximum value at the fixed support. For all wood column cases, the dynamic shear force is greater than the static shear force at the impact point, and C2P2 gives the maximum shear force amplification factor at the fixed end.

For bending moment, case C1P2 always gives the maximum moment at the fixed end. For concrete column and steel column, in all cases except C2P2, the minimum value of the dynamic bending moment occurs around 1/4 the column length and its value is around 0.25. This is because the moment at 1/4 the column length vanishes if a concentrated load is applied at mid-span of the column. However for wood column, the minimum value of \( R_b \) is around 0.2 due to the complexity of harmonics. It can be concluded that the stiffer the column, the higher the bending, and multiple impacts always magnify the dynamic shear force and bending moment at later impacts.
(a) Concrete column

(b) Steel column
Figure 48: Maximum shear force (left) and bending moment (right) amplification factors for axial impact

5.4.2.2 Transverse Impacts

Figure 49 shows the amplification factor of maximum shear force and bending moment. The symbols in legends are: cC1 is short concrete column, cC2 is long concrete column, sC1 is short steel column, sC2 is long steel column, wC1 is short wood column, and wC2 is long wood column.

It is clear from Figure 49 that stiffer columns always give higher dynamic amplification factors for moment and shear. This is due to the high frequency vibration. In addition, for cases involving multiple impacts, the dynamic amplification factor increases always at later impacts. Short concrete columns always give the largest dynamic shear force and bending moment. For all cases, the dynamic shear force is greater than the static shear force at the impact point. Maximum shear force amplification factor is attained for short concrete column at the fixed support.

In all cases except wood, the dynamic bending moment is less than the static bending moment for long columns. Maximum bending occurs at the fixed end for short concrete column with a value of 1.88. The minimum value is attained around 1/4 the column length and its value
ranges between 0.2 and 0.55. Table 6 to Table 9 show the dynamic amplification factors for different impact scenarios.

(a) Shear force
Figure 49: Maximum amplification factors for transverse impact

Table 6: Maximum dynamic amplification factors for shear force for axial impact

<table>
<thead>
<tr>
<th>Case</th>
<th>C1P1</th>
<th>C1P2</th>
<th>C2P1</th>
<th>C2P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete Column</td>
<td>2.44</td>
<td>2.53</td>
<td>1.22</td>
<td>1.73</td>
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<tr>
<td>Steel Column</td>
<td>1.74</td>
<td>3.14</td>
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<tr>
<td>Wood Column</td>
<td>2.75</td>
<td>2.61</td>
<td>2.03</td>
<td>2.29</td>
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</tbody>
</table>
Table 7: Maximum dynamic amplification factors for bending moment for axial impact

<table>
<thead>
<tr>
<th>Case</th>
<th>C1P1</th>
<th>C1P2</th>
<th>C2P1</th>
<th>C2P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete Column</td>
<td>0.73</td>
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<td>0.44</td>
<td>0.93</td>
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<tr>
<td>Steel Column</td>
<td>1.46</td>
<td>1.99</td>
<td>0.64</td>
<td>1.82</td>
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<tr>
<td>Wood Column</td>
<td>0.69</td>
<td>1.32</td>
<td>0.45</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Table 8: Maximum dynamic amplification factors for shear force for transverse impact

<table>
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<tr>
<th>Case</th>
<th>3.6 m column</th>
<th>6 m column</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete Column</td>
<td>4.71</td>
<td>3.68</td>
</tr>
<tr>
<td>Steel Column</td>
<td>4.15</td>
<td>2.63</td>
</tr>
<tr>
<td>Wood Column</td>
<td>3.38</td>
<td>2.20</td>
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</table>

Table 9: Maximum dynamic amplification factors for bending moment for transverse impact

<table>
<thead>
<tr>
<th>Case</th>
<th>C1</th>
<th>C2</th>
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</thead>
<tbody>
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<td>Concrete Column</td>
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<td>0.89</td>
</tr>
<tr>
<td>Steel Column</td>
<td>1.46</td>
<td>0.88</td>
</tr>
<tr>
<td>Wood Column</td>
<td>0.82</td>
<td>0.59</td>
</tr>
</tbody>
</table>

5.5 Summary

Additional scenarios were investigated for multiple impacts. In the first part, the linear response of a wood pole hitting axially a fixed-end column is studied. The material of the column can be concrete, steel, or wood and its length is 3.6 m or 6 m. The length of the pole is 3 m or 6 m. The impact occurs at the mid-span of the column. In the second part, the linear response of a 9 m wood pole hitting transversely a 6 m column is studied. The column end
boundary conditions can be free or pinned. The impact also occurs at the mid-span of the pole and the column. All the results presented in this chapter are obtained using FEM solution. The following conclusions may be made.

1. For a wood pole hitting axially a concrete column, the collision event is characterized by a single impact for all cases. The reason for this is that the high stiffness of the column causes the column to vibrate with high frequencies and small amplitudes at the impact point. Most of the kinetic energy of the pole is transferred into axial strain energy of the pole which attains its maximum value when the axial wave in the pole hits the far end of the pole. The mass-averaged rebound velocity of the pole is approximately 80% of the initial velocity.

2. For a wood pole hitting axially a steel column, only the case of a 9 m pole and a 6 m column exhibits multiple impacts consisting of two contact phases. The maximum impact force occurs in the second contact phase and is almost 1.6 times the instantaneous peak force. For the other cases having a single impact, most of the deformation in the column is due to shear and most of the kinetic energy of the pole is transferred into strain energy of the pole. For the case of multiple impacts, most of the energy is transferred into axial strain energy of the pole shortly after the first impact. After that, the bending energy and the translational kinetic energy of the column absorb most of the kinetic energy of the pole.

3. For the case of a wood pole hitting axially a wood column, all the cases experience multiple impacts. The maximum force occurs near the end of the final separation and reaches up to 1.9 times the instantaneous impact force. The axial strain energy of the pole is very small throughout the entire collision event. Most of the kinetic energy of the pole is transferred into bending energy of the column. The mass-averaged rebound velocity of the pole ranges between 70% and 90% of the initial velocity.

4. For a 9 m wood pole hitting transversely a 6 m wood column, the maximum force for the free-end column is almost equal to peak instantaneous impact force. However, for the case of pinned-end column, the impact force reaches up to 1.48 times the peak instantaneous force. For the case of a free-end column, most of the energy of the column is translational kinetic energy and the column and the pole move in the same direction of
the initial velocity throughout the entire collision event. On the other hand, most of the kinetic energy of the pole is transferred into bending energy of the pole and the column. The impulse at the final separation is 0.6 times the initial momentum of the pole in the case of free-end column. However, it reaches up to 1.9 times the initial momentum of the pole for the pinned-end column.

5. For axial and transverse impacts, cases involving stiffer columns and higher inertia of the pole always give higher shear force and bending moment dynamic amplification factors. In addition, for cases involving multiple impacts, the dynamic effect is magnified always at later impacts. Hence, cases of short fixed-end column and long pole give the maximum dynamic shear force for single impact and give the maximum bending moment for single and multiple impacts. However, cases of long column and long pole give the maximum dynamic shear force.
Chapter 6

Conclusion

The elastic impact of a translating flexible pole is studied herein. Three scenarios are considered: 1) transverse impact against a rigid stop, 2) longitudinal impact against a flexible column, and 3) transverse impact against a flexible column. Based on Timoshenko beam theory, an analytical solution method using mode superposition for the coupled spring-pole or column-pole system is presented. Any physical set of boundary conditions can be accommodated for the pole and the column.

For all cases involving axial impacts, the maximum initial impact force is governed by the local shear deformation in the column and the axial deformation in the pole. However, for transverse impacts, the maximum initial impact force is governed by the local shear deformation in the pole and the column. A simple formula for the maximum initial force is derived and shown to be quite accurate. In no case is the total mass of the pole significant to the initial peak force. Indeed, based on Euler-Bernoulli beam theory the initial impact force is unbounded as the spring stiffness increases whereas Timoshenko beam theory has a clear limiting value for the initial impact force. The impact duration depends on the wave propagation in the pole or the column.

6.1 Response of a Beam Hitting Transversely a Stop

An analytical solution for the linear response of a Timoshenko beam impacting a stop modeled as a spring has been presented. The solution, which is based on modal superposition, admits contact at an arbitrary point within the beam span, as well as beam rotational as well as translational initial velocities. As such, the present solution is a generalization as compared to previous solutions. It has been shown that the initial impact is dominated by shear behavior, and that for stiff stops, Euler-Bernoulli beam theory can significantly overestimate the initial impact force and underestimate the contact duration. Indeed, based on Euler-Bernoulli beam theory the impact force grows without bounds as the stop stiffness increases, whereas Timoshenko beam theory shows the impact force is bounded. As a general rule, it is recommended that Timoshenko beam theory rather than Euler theory be used for impact studies. Parameter studies have shown
that the impact force time history is controlled primarily by the ratio of the stop stiffness to the beam shear stiffness. For homogeneous beams, the sectional rotational inertia has little effect on the impact forces.

### 6.2 Beam Response to Longitudinal Impact by a Pole

An analytical solution for the linear impact response of a pole impacting a column, or a beam impacting a flexible axial stop, has been presented. The solution is based on Timoshenko beam theory and uses modal superposition. Contact can occur at an arbitrary point within the column span. The following conclusions may be made.

1. The initial peak impact force can be predicted by a simple formula, Eq. (3.15). In the case of stiff columns, the pole properties govern the force; in the case of a stiff pole, the column properties govern. In no case is the total mass of the projectile significant to the initial peak force.
2. The impact duration is governed by either the longitudinal wave speed in the pole (for stiff columns) or the shear wave speed in the column.
3. For cases in which the column properties are significant, the initial response is dominated by shear behavior, and Timoshenko beam theory rather than Euler-Bernoulli beam theory should be used. Indeed, based on Euler-Bernoulli beam theory the initial impact force is unbounded as the stop stiffness increases, whereas Timoshenko beam theory has a clear limiting value for the initial impact force.
4. The rotary inertia does not affect the initial peak impact force but it can have an effect on the impact force time history after waves in the beam return to the impact point.
5. The energy plots provide a vivid picture of the complex interactions and energy exchange taking place during the impact. For instance, of direct relevance to the present work, it explains the importance of the shear mode at the beginning of the impact to set the impact force and also depicts the relevance of the various energy exchanges in setting later spikes in the force as well as the impact duration. The energy exchange also shows the importance of the inertia of the column in absorbing a significant part of the initial kinetic energy of the pole.
6.3 Transverse Impact of a Horizontal Beam on a Vertical Column

The linear response of a beam hitting transversely a fixed-fixed column is studied in this paper. The results presented are for a 9 m wood pole hitting transversely concrete, steel, and wood columns having lengths of 3.6 m and 6 m. The pole moves with an initial velocity and hits laterally the column at midspan. Multiple impacts were considered and the whole collision event is divided into contact and separation phases. Results from the analytical solution were used to verify the finite element solution. In all cases excellent matches were obtained. The following conclusions may be made.

1. The initial peak force is predicted well by Eq. (4.13). However, the maximum impact force can reach up to 1.6 times the initial peak force.
2. Variation in initial peak impact force depends mainly on the mass ratio of the column and the pole.
3. The instantaneous impact force is dominated by shear deformation either in the pole or in the column, or both.
4. For most of the cases, there is a high frequency pounding after a fairly large separation phase. However, for the most flexible (long wood) column, the durations of all contact and separation phases are small due to the high frequency pounding between the pole and the column.
5. For concrete and steel columns, the sum of the bending strain energy and the kinetic translational energy in the pole constitutes almost 90% of the total energy during the collision event.
6. For the soft (wood) column, a significant part of the total energy is absorbed by the bending strain energy and translational kinetic energy of the column during the collision event.
7. For the most rigid (short concrete) column, only 1.5% of the initial energy is transferred to the column. However, for the most flexible (long wood) column, almost 7.2% of the initial energy is transferred to the column. In this case, at the time of separation most of the energy transferred to the column is absorbed by bending strain energy at final separation.
8. The total impulses for the concrete and steel columns are almost 1.6 times the initial momentum of the pole. For the wood columns, the total impulse is almost 1.8 times the initial momentum of the pole. This indicates that the final mass-averaged rebound velocity of the pole is almost 60% of the initial velocity for the concrete and steel columns, and 80% of the initial velocity for the wood columns.

6.4 Additional Scenarios

Additional scenarios were investigated for multiple impacts. In the first part, the linear response of a wood pole hitting axially a fixed-end column is studied. The material of the column can be concrete, steel, or wood and its length is 3.6 m or 6 m. The length of the pole is 3 m or 6 m. The impact occurs at the mid-span of the column. In the second part, the linear response of a 9 m wood pole hitting transversely a 6 m column is studied. The column end boundary conditions can be free or pinned. The impact also occurs at the mid-span of the pole and the column. All the results presented in this chapter are obtained using FEM solution. The following conclusions may be made.

1. For a wood pole hitting axially a concrete column, the collision event is characterized by a single impact for all cases. The reason for this is that the high stiffness of the column causes the column to vibrate with high frequencies and small amplitudes at the impact point. Most of the kinetic energy of the pole is transferred into axial strain energy of the pole which attains its maximum value when the axial wave in the pole reaches the far end of the pole. The mass-averaged rebound velocity of the pole is approximately 80% of the initial velocity.

2. For a wood pole hitting axially a steel column, only the case of a 9 m pole and a 6 m column exhibits multiple impacts, and in that case there are two contact phases. The maximum impact force occurs in the second contact phase and is almost 1.6 times the instantaneous peak force. For the other cases having a single impact, most of the deformation in the column is due to shear and most of the kinetic energy of the pole is transferred into strain energy of the pole. For the case of multiple impacts, most of the energy is transferred into axial strain energy of the pole shortly after the first impact.
After that, the bending energy and the translational kinetic energy of the column absorb most of the kinetic energy of the pole.

3. For the case of a wood pole hitting axially a wood pole, all the cases experience multiple impacts. The maximum force occurs near the end of the final separation and reaches up to 1.9 times the instantaneous impact force. The axial strain energy of the pole is very small throughout the entire collision event. Most of the kinetic energy of the pole is transferred into bending energy of the column. The mass-averaged rebound velocity of the pole ranges between 70% and 90% of the initial velocity.

4. For a 9 m wood pole hitting transversely a 6 m wood column, the maximum force for the free-end column is almost equal to the peak instantaneous impact force. However, for the case of a pin-end column, the impact force reaches up to 1.48 times the peak instantaneous force. For the case of a free-end column, most of the energy of the column is translational kinetic energy and the column and the pole move in the same direction of the initial velocity throughout the entire collision event. On the other hand, most of the kinetic energy of the pole is transferred into bending energy of the pole and the column. The impulse at the final separation is 0.6 times the initial momentum of the pole in the case of the free-end column. However, it reaches up to 1.9 times the initial momentum of the pole for the pin-end column.

5. For axial and transverse impacts, cases involving stiffer columns and higher inertia of the pole always give higher shear force and bending moment dynamic amplification factors. In addition, for cases involving multiple impacts, the dynamic effect is magnified always at later impacts. Hence, cases of short fixed-end column and long pole give the maximum dynamic shear force for single impact and give the maximum bending moment for single and multiple impacts. However, cases of long column and long pole give the maximum dynamic shear force.
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ASCE (2010). “Minimum design loads for buildings and other structures.” ASCE/SEI 7-10, American Society of Civil Engineers.


