AN INTERFACING STRATEGY FOR FLUID-STRUCTURE INTERACTION
WITH APPLICATION TO LINEAR HYDROELASTICITY

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ABSTRACT

To improve the accuracy of the numerical procedure in hydroelastic analysis, two areas are studied in this work. One is the interface methods used to couple the fluid and the structural meshes. The other is the hydrostatic stiffness for use in hydroelastic analysis of flexible floating structures.

Nonlinear, time-domain hydroelastic analysis of flexible offshore structures requires that the structural motion be transferred to the fluid model and the resulting fluid pressure at the fluid-structure interface be transferred from the fluid model to the structure. When the structural mesh and the fluid mesh describe two distinct three-dimensional surfaces, the transfer of the displacement and pressure is both difficult and non-unique. A new transfer methodology based on the variational-based smoothing element analysis (SEA) technique is presented. The displacement transfer uses the original formulation of the SEA, although the application of the procedure to displacement transfer is new. For energy conservation during the reverse pressure transfer, SEA is modified. The transfer method is tested by examining the performance of three floating rigid bodies. Application of the methodology to flexible bodies is also presented. The numerical results show that the method works very well.

The formulation of the hydrostatic stiffness for linear rigid body hydrodynamics is well-known. An explicit formulation for an analogous hydrostatic stiffness in linear hydroelasticity, which is applicable to both rigid body and flexible displacement, is not well-known. Three such formulations have been proposed previously in the literature, none of which is quite correct. An explicit formulation for the complete hydrostatic stiff-
ness matrix for flexible structures, for use in linear hydroelastic analysis, is derived based on a consistent linearization of the generalized external and internal forces. The symmetry of the present formulation for a floating structure is proven analytically, and the unsymmetry of the hydrostatic stiffness for individual finite elements is discussed.
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CHAPTER 1
INTRODUCTION

1.1 Overview

The numerical simulation of fluid-structure interaction phenomenon, which is a difficult problem both mathematically and computationally, occurs in many scientific and engineering applications. Hydroelasticity, for example, requires the coupling of the hydrodynamic and structural responses. The structure and the fluid have quite different characteristics, as do the equations governing their behaviors. This feature results in special complexities in hydroelastic analysis. To improve the accuracy of the numerical procedure in hydroelastic analysis, two areas have been studied in this work. One is the interface methods used to couple the fluid and the structural interface. The other is the hydrostatic stiffness for use in hydroelastic analysis of flexible floating structures.

1.1.1 Interface methods

For historical and practical reasons, different analysis methodologies are often used for the fluid and the structure when they are analyzed separately. In structural analysis, for all but the most geometrically simple structures, the finite element method is dominant. For the fluid, finite element, finite difference and Green function/boundary element methods are often used. For the combined fluid-structure problem, one can adopt either a ‘closely-coupled’ strategy or a ‘loosely-coupled’ strategy [1, 2]. The ‘closely-coupled’ approach involves a single model for both the fluid and structure. In the ‘loosely-coupled’
approach, separate models are used that interface through a transfer of data. Hence, those numerical methodologies that have been developed for two different ‘homogeneous’ systems (fluid and structure) can be used virtually unchanged for the coupled heterogeneous system. In addition, initial validation and verification of the models and implementations can be carried out independently. This reduces substantially the validation and verification required for the much more complex coupled system [3]. Therefore, the loosely-coupled approach is the most common. For example, in linear hydroelasticity in the frequency domain, a common solution strategy involves the transfer of the finite element mode shapes to the fluid model. In nonlinear time domain hydroelasticity, not only is the structural motion transferred to the fluid model, but also the resulting fluid pressures (or at least nodal loads) on the wetted surfaces are transmitted from the fluid model to the structure.

If the structural and fluid meshes of the wetted surface are identical, i.e., there is a 1-1 mapping of fluid and structural elements, it is relatively straightforward to design a transfer strategy. In practice, the requirements to generate the discretized models of two disciplines are subject to different engineering considerations and the models are often designed by different analysts. Therefore, the structural model is often different from the fluid model. This gives rise to the interfacing problem of transferring the data between these two models. For practical structures, the wetted surfaces are usually curved, and the structural and fluid meshes rarely if ever describe the same wetted surface. That is, each mesh only approximates geometrically the physical interface, and the two approximations are not the same. As a result, neither the displacement transfer nor the pressure transfer has
A unique mathematical solution, and an explicit robust transfer strategy is important.

A number of interfacing methods are used. The methods can be categorized as either energy non-conservative or conservative methods. The former do not conserve the work done by the fluid pressures/loads when they are transferred to the structural mesh, and they need fine meshes for both the fluid and structure to give accurate results [4]. In the literature, there are few energy conservative methods that have been applied to 3D bodies. Furthermore, all such methods attempt to transfer equivalent nodal loads to the structural model rather than the pressure field; see, e.g., [1, 4-9].

The new transfer strategy presented in this study is an energy conservative method, which is based on the variational-based smoothing element analysis (SEA) method. SEA has been developed primarily for recovery of finite element stresses. However, SEA can be used to ‘smooth’ any data, and in fact the basic procedure was developed originally to filter, smooth and interpolate discrete experimental data. The displacement transfer of this strategy is a reasonably straightforward application of SEA, although the application of the procedure to displacement transfer is new. In the reverse, pressure transfer, this strategy attempts to transfer the pressure field to the structural model with high fidelity. To ensure conservation of energy, the original functional in SEA is modified to conserve the work done by hydrodynamic pressures when they are transferred to the structural mesh. Three modifications of SEA to impose such an equal-energy criterion are considered.

The proposed strategy offers several appealing features, including a robust mapping scheme, energy conservation properties, acceptance of relatively large geometrical difference between the fluid mesh and the structural mesh, consistent nodal forces and moments
computed from the smoothed pressure contour, and improved accuracy of the results.

1.1.2 The hydrostatic stiffness of flexible floating structures for linear hydroelasticity

The evaluation of the linear wave-induced motion of a floating structure about an initial, mean equilibrium position is an important field of hydrodynamics. For large, inertia-dominated, rigid bodies, a common approach is to use linear potential theory to determine the hydrodynamic forces, and the motion is then determined in the frequency domain. In the linearized equations of motion, the inertial forces are represented by an added mass matrix, which is combined with the structural mass matrix, and the damping forces are represented by a hydrodynamic damping matrix. There also are forces that depend on the structural displacements; they represent the changes in the hydrostatic pressure forces and the structural forces when the structure is displaced from the equilibrium position. In linear rigid body hydrodynamics, the changes in these forces are characterized by the restoring force coefficients, the formulation for which is well known. These coefficients form the ‘hydrostatic’ stiffness matrix.

In linear hydroelasticity based on linear potential theory, this analysis methodology is extended to deformable bodies. An assumed-mode/generalized coordinate approach is often used to reduce the very large number of radiation potentials which would otherwise be required. The extension of the formulation of the added mass and hydrodynamic damping matrices in terms of assumed modes is relatively straightforward and well-known [10]. An explicit formulation for the ‘complete’ hydrostatic stiffness matrix does not appear to be as well-known. By ‘complete’ we mean that, analogous to the rigid body case, it should
include all first order variations in the forces associated with the initial equilibrium configuration. As such, it will contain as a special case the hydrostatic stiffness used in rigid body analysis and which is required for rigid body motion of the deformable body. In the linearized equations of motion, this matrix will be combined with the structural stiffness matrix that depends on the material properties. Three explicit formulations have been proposed in the literature, none of which is quite correct and all of which result in an unsymmetric stiffness matrix.

A complete explicit formulation is presented herein that distinguishes itself from the other three previous formulations. The formulation has the required symmetry, and it also allows an alternative derivation of the geometric stiffness matrix based on effective tension that is often applied to submerged frame elements in offshore engineering. This concept is extended to plate elements. The formulation is consistent with the one used in linear, rigid body hydrodynamics, and it contains that formulation as a special case. The new formulation not only results in a symmetric hydrostatic stiffness matrix, it also reveals the unsymmetric contributions from individual elements that are used to model a structure. The formulation will likely be of most interest to those who wish to extend existing linear potential theory hydrodynamic codes for rigid body analysis to deformable bodies.

1.2 Objective and Scope of Work

This study has two primary objectives. The first objective is to propose a new strategy for the displacement and pressure transfer in fluid-structure interaction problems. The second objective is to derive an explicit formulation for the complete hydrostatic stiffness
for use in hydroelastic analysis of flexible floating structures.

The dissertation is organized as follows. In Chapter 2, a review of numerical fluid/
structure interface methods in both aeroelasticity and hydroelasticity is presented. The
basic formulation of SEA/PDLS (Smoothing Element Analysis/Penalized Discrete Least-
Squares) is also reviewed. Chapter 3 gives algorithms for the transfer strategy and the
detailed derivation of three pressure transfer methods. The performance of these methods
is evaluated with several examples in Chapter 4. The application to flexible bodies is pre-
sented in Chapter 5. In Chapter 6, the formulation for the complete hydrostatic stiffness
for flexible floating structures at rest in calm water is presented. Several issues that are sig-
nificant for practical implementation are discussed, and several examples are presented to
illustrate the application of the formulation. Finally, conclusions and recommendations are
given in Chapter 7.
CHAPTER 2

REVIEW OF TRANSFER STRATEGIES AND SEA

2.1 General Comments

This chapter provides a technical overview of displacement and pressure/force transfer strategies in both hydroelasticity and aeroelasticity. The consistent interpolation based method, virtual surface method, and Farhat’s conservative algorithm are reviewed in detail. Smoothing element analysis, on which the transfer strategy proposed in this study is based, is also reviewed in detail.

2.2 Review of Transfer Strategies

2.2.1 Overview

Much of the published work on transfer strategies has been done in aeroelasticity rather than hydroelasticity. In the early 1970s, Harder and Desmarais [11] developed an infinite plate spline (IPS) method in aeroelastic analysis, which is the groundwork for two-dimensional interfacing methods. The basic idea is to use the small deflection equation of an infinite plate for interpolating a function of two variables. This method was originally developed for interpolating wing deflections and computing slopes for aeroelastic calculations. Later, subsequent experience with IPS indicated that extrapolations to the edges of the platform from the interior structural grid points don’t always appear to be reliable [12].

A node-to-element method has been used [1, 4-6] based on four-node isoparametric finite elements and inverse isoparametric mapping. The idea of inverse isoparametric map-
ping is to find a local coordinate \((\xi, \eta)\) from the information given in the global coordinates \((x, y)\). The displacements at structural nodes can be interpolated to fluid points using isoparametric interpolation. The fluid nodal force at a given fluid point \((x, y)\) can be proportionately distributed to structural nodes using \((\xi, \eta)\) values. This approach was successfully applied for a wing-body configuration \([6]\). However, the application of these methods to curved three-dimensional bodies is unclear. Also, the structural mesh is limited to four-node or eight-node isoparametric finite elements.

Other schemes include one in which every Gauss point of the structural elements is paired with a Gauss point of a fluid panel \([7]\). Chen and Jadic \([8]\) presented an alternative approach based on the structural boundary element concept, the assumption in which is that the structural nodes are located within or on the surface defined by the fluid surface grid. The method involves a system of equations with a full unsymmetric coefficient matrix, which lessens its attractiveness. In hydroelastic analysis, a simple average method has been used to transfer structural nodal displacements to the fluid model. The required displacement on the fluid model is obtained by averaging the displacements of those structural nodes that are closest to the required fluid panel point, see e.g., \([13]\).

The above methods are known as energy non-conservative methods; i.e., they don’t conserve the work done by the fluid pressures and they need fine meshes for both the fluid and structure to give accurate results \([4]\). There are few energy conservative methods with application to curved 3D bodies in the literature. Two energy conservative methods, a virtual surface method \([4, 12]\) and a conservative algorithm presented by Farhat et al. \([7, 14]\), will be discussed subsequently.
2.2.2 Consistent interpolation based method

The finite element representation of the nodal loads induced by the fluid pressure, \( p \), acting on the structural element \( e \) can be written as

\[
    f_i^e = \int_{\Omega_e^s} N_i^T (-p \mathbf{n}) \, ds
\]

where \( \Omega_e^s \) denotes the structural element domain, \( \mathbf{n} \) denotes the normal to the structural surface, and \( N_i \) is the interpolation function for the displacement field associated with the node \( i \) of the element \( e \).

In the consistent interpolation based method [7], numerical quadrature is used to evaluate the integral in equation (2.1):

\[
    f_i^e = \sum_{g=1}^{n_g} w_g N_i(\mathbf{X}_g)(-p(\mathbf{X}_g)\mathbf{n})J_g
\]

where \( w_g \) is the weight of the Gauss point \( \mathbf{X}_g \), which is a sampling point defined by the Gauss quadrature rule; \( n_g \) is the number of Gauss points used, and \( J_g \) is the Jacobian determinant.

It is necessary to evaluate the fluid pressure at the Gauss points of the structural elements. The approach [7] is to pair every Gauss point with a fluid panel. Note that the nodal loads calculated by equation (2.2) include both forces and moments. Moments can be important in many fluid/structure applications [7]. However, it is difficult to achieve the pairing criteria when the geometry of the interface becomes complex. It is also clear that this approach does not guarantee that the sum of the discrete loads on the structural surface are equal to the sum of the fluid loads on the fluid surface [7].
2.2.3 **Virtual surface method**

The virtual surface method introduces a virtual surface between the fluid mesh and the structural mesh, which is discretized by a number of finite elements. The discretization of the virtual surface is not necessarily the same as the structural discretization. Three displacements for each node of the virtual surface are considered: one transverse displacement and two rotations (about the \(x\)-axis and \(y\)-axis). Fluid points and structural nodes are distributed on the virtual surface and may not have a regular layout.

![Figure 2.1 VS method [4]](image)

The virtual surface method is based on a mapping matrix developed by Appa [9] and Appa et al. [12]. This method is illustrated in Figure 2.1. Let \( \mathbf{q}_a \) and \( \mathbf{q}_s \) denote the displacement vector at the fluid points and the displacement vector at the structural nodes, respectively. Forcing the deformed virtual surface to pass through the given data points of the deformed structure results in the constraint equations

\[
\mathbf{q}_s = \Psi_s \mathbf{q} \tag{2.3a}
\]

\[
\mathbf{q}_a = \Psi_a \mathbf{q} \tag{2.3b}
\]
in which $\mathbf{q}$ is the displacement vector at the nodes on the virtual surface, $\psi_s$ is a displacement mapping from the virtual surface to the structural nodes, and $\psi_a$ is a displacement mapping from the virtual surface to the fluid points (see [4, 9, 12] for more details).

The penalty method and a least squares technique are used to enforce the constraint equation (2.3a), and the ‘equilibrium’ of the virtual surface is given by

$$[\mathbf{K} + \alpha\psi_s^T\psi_s]\mathbf{q} = \alpha\psi_s^T\mathbf{q}_s$$

(2.4)

in which $\alpha$ is a penalty parameter and $\mathbf{K}$ is the ad hoc stiffness of a free-free plate, which is added for stability.

The displacement vector at the fluid points $\mathbf{q}_a$ can be expressed in terms of the displacement vector at the structural nodes $\mathbf{q}_s$. Substitution of the solution of $\mathbf{q}$ in equation (2.4) into (2.3b) results in

$$\mathbf{q}_a = \mathbf{T}\mathbf{q}_s$$

(2.5)

where

$$\mathbf{T} = \psi_a\{\alpha^{-1}\mathbf{K} + \psi_s^T\psi_s\}^{-1}\psi_s^T$$

From the principle of virtual work, the structural nodal force vector, $\mathbf{R}_s$, can be obtained as

$$\mathbf{R}_s = \mathbf{T}^T\mathbf{R}_a$$

(2.6)

where $\mathbf{R}_a$ is the force vector on the fluid mesh.

Note that the nodal loads at each structural node calculated by equation (2.6) include one transverse nodal force and two moments. This method transfers fluid forces accurately and is energy-conservative [4]. However, application of this method to 3D bodies is not
mentioned. Also, this approach involves a system of linear equations that becomes ill-conditioned for irregular structural meshes [5].

2.2.4 Farhat’s conservative method

Farhat et al. [7] introduced a method based on energy conservation. The displacement field on the fluid surface, $u_f$, can be expressed as

$$u_f = D\hat{u}_f$$

(2.7)

in which $D$ is the interpolation function for the displacement field over the fluid surface, and $\hat{u}_f$ is the displacement vector at the fluid points.

The displacement vector at the fluid points, $\hat{u}_f$, can be expressed in terms of the displacement vector at the structural nodes, $\hat{u}_s$, as

$$\hat{u}_f = N\hat{u}_s$$

(2.8)

in which $N$ is the interpolation function of the structural elements evaluated at the corresponding fluid points.

If virtual work is used to ensure the pressures acting on the structural and fluid models do the same work, the following structural nodal forces are obtained:

$$\mathbf{F} = N^T \int_{\Gamma_f} D^T (-p\mathbf{n}) d\Gamma$$

(2.9)

in which $\mathbf{n}$ denotes the normal to the fluid surface $\Gamma_f$, and $p$ is the fluid pressure on the fluid mesh.

The mapping scheme proposed in this method is to pair each fluid point on the fluid surface with the closest structural element, and to determine the natural coordinates of the
fluid point on the structural mesh so that the displacements of the fluid points can be obtained by interpolating the structural nodal displacements using the interpolation function of the structural elements. Note that this mapping scheme gives rise to several problems, which will be discussed in Section 3.2. In addition, it is often difficult to compute the natural coordinates of those fluid points on the structural mesh for known global coordinates, and most FEA codes do not have this capability.

### 2.2.5 Conclusions

From the aforementioned methods, it can be concluded that a transfer strategy includes two components. First, a mapping strategy is required to locate pressure points from the fluid mesh on the structural mesh and to locate the structural nodes on the fluid mesh. Second, interpolation is required to obtain the value of the transferred data at the required points, e.g., the structural displacements at the fluid nodes. Mapping from 2D bodies to 2D bodies is straightforward. The major difficulty for transfer strategies is when the fluid and structural meshes define two distinct surfaces. There are several mapping schemes discussed in the above methods. None of these methods are robust and explicit when applied to 3D bodies. Furthermore, all such methods attempt to transfer equivalent nodal loads to the structural model rather than the pressure field. As a result, a new energy conservative transfer strategy is needed.

### 2.3 Review of Smoothing Element Analysis (SEA)

#### 2.3.1 Overview

The transfer strategy proposed in this work is based on a modified smoothing ele-
ment analysis (SEA) methodology. SEA is a post-processing recovery procedure that has been developed over the last several years by Tessler, Riggs and coworkers [15-18]. SEA is based on a penalized discrete-least-square (PDLS) variational principle which combines discrete-least-squares and a penalty constraint in a single variational form. It is basically a finite element method for recovering a higher-order accurate, $C^1$ continuous stress field from discrete stress data, which are extracted from the underlying discontinuous finite element stress field. However, SEA can be used to ‘smooth’ any data, and in fact the basic procedure was developed originally to filter, smooth and interpolate discrete experimental data. SEA has proven to be robust and capable of recovering a superconvergent stress field of significantly higher accuracy than the underlying stress field. The two dimensional formulation, which is of interest herein, is reviewed subsequently. More detailed information on SEA can be found in [15-18].

2.3.2 Problem definition

The problem domain is denoted by $\Omega = \{x \in R^2\}$, where $x = \{x, y\}$, is a position vector in Cartesian coordinates. The discrete data, $p_q^f$, are defined at $x_q, q = 1, 2, \ldots, N$ in $\Omega$. (The superscript $f$ indicates the data are coming from the fluid mesh; if the data are coming from the structural mesh, a superscript $s$ will be used.) The smoothed field, $p(x)$, is to be recovered from $p_q^f$ via a PDLS variational formulation. The error functional only involves scalar quantities, and therefore, if the underlying data are tensor quantities, each component is recovered independently. To minimize the error functional, the finite element methodology is adopted and therefore, the problem domain $\Omega$ is discretized with
Finite element interpolation functions are used to obtain $C^0$-continuity for the primary variables $p$, and the independent quantities $\theta_i$, $i = x, y$, the interpretation of which will soon become clear. The error functional is [17]

$$
\Phi = \frac{1}{N} \sum_{q=1}^{N} (p_q^f - p(x_q))^2 + \alpha \sum_{e=1}^{N_p} \left[ \int_{\Omega_e} [(p_{x} - \theta_x)^2 + (p_{y} - \theta_y)^2] d\Omega \right]
$$

in which $\alpha$ and $\beta$ are dimensionless penalty parameters; $\Omega_e$ is the area of element $e$; and the comma indicates partial derivative. The first term in equation (2.10) represents the error between the smoothed data and the discrete sample data. The second term represents a penalty functional that, for sufficiently large $\alpha$, enforces the derivatives of the smoothed data field to equal the $\theta_j$. Because $\theta_j$ are interpolated with continuous functions, the smoothed field is (nearly) $C^1$ continuous. Results have shown that the method is remarkably robust with respect to the value of $\alpha$, and a wide range of values can be used [16]. The third, optional, term is a ‘curvature’ control term that provides stability should the discrete input data be insufficient, because either there are too few data points or their spatial distribution is poor, to define uniquely the smoothed field for the given smoothing mesh. Typically, $\beta$ should be a very small number (zero if the input data are known to be sufficient).

### 2.3.3 Smoothing element

A 3-node, triangular, two-dimensional finite element based on equation (2.10) has been discussed in detail in [15, 16], and therefore the basic features of the element are reviewed only briefly. The 3-node triangular element has three degrees-of-freedom per
node: \( p, \theta_x, \) and \( \theta_y \). Interpolation functions, originally developed for a Mindlin plate element [19, 20], are such that the interpolation for \( p \) is quadratic and the interpolations for \( \theta_i \) are linear. In particular, the interpolations for element ‘e’ are:

\[
p^e = P^e p + Q_x \theta^e_x + Q_y \theta^e_y = N^e d^e
\]

(2.11a)

\[
\theta^e_i = P^e \theta^e_i, \ i = x, y
\]

(2.11b)

in which \( P, Q_x, \) and \( Q_y \) are row vectors of interpolation functions given below; \( p^e \) is a 3x1 vector of nodal values of \( p; \) \( \theta^e_x \) and \( \theta^e_y \) are 3x1 vectors of nodal derivatives of \( p; \) and \( N \) and \( d^e \) are composite vectors. The interpolation functions in equations (2.11a) and (2.11b) are simplest when written in terms of the area-parametric coordinates \( \zeta = [\zeta_1, \zeta_2, \zeta_3] \), which are defined in terms of the nodal Cartesian coordinates \( x_k, y_k \) and the element area \( A \):

\[
\zeta_i = \frac{(c_i + b_i x + a_i y)}{2A}, a_i = x_k - x, b_i = y_j - y, c_i = x_j y_k - x_k y_j
\]

(2.12a)

The subscripts are given by the cyclic permutation of \( i = 1, 2, 3; j = 2, 3, 1; \) and \( k = 3, 1, 2. \)

The interpolation functions are then given by

\[
P_i = \zeta_i, \ Q_{xj} = \frac{1}{2} (a_k \zeta_i \zeta_j - a_j \zeta_i \zeta_k), \ Q_{yj} = \frac{1}{2} (b_j \zeta_i \zeta_k - b_k \zeta_i \zeta_j)
\]

(2.12b)

The element stiffness matrix has three components, resulting from the three terms of the functional of equation (2.10):

\[
K^e = K^e_e + K^e_\alpha + K^e_\beta = \frac{1}{N} \sum_{q=1}^{n^e} N_q^T N_q + \alpha \int_{\Omega^e} B_{\alpha}^T B_{\alpha} d\Omega + \beta \int_{\Omega^e} B_{\beta}^T D B_{\beta} d\Omega
\]

(2.13)

in which
and where \( n^e \) is the number of sample points within element \( e \), and \( N_q = N(x_q) \). Note that the sample points have been renumbered on an element basis in equation (2.13). The consistent element ‘load’ vector (right-hand side) is

\[
F^e = \frac{1}{N} \sum_{q=1}^{n^e} p_q^f N_q^T
\]  

(2.15)

The usual finite element assembly procedure is used to obtain the system equations from the element matrices, resulting in

\[
Kd = F
\]  

(2.16)

From equation (2.13) it is clear that while \( K \) depends on \( x_q \), it is independent of the \( p_q^f \). Hence, while equation (2.16) must be solved for each component (e.g., each displacement component), the individual components merely represent different load cases.

It is interesting to note that SEA is closely related to the virtual surface method discussed earlier.
CHAPTER 3

FORMULATION OF TRANSFER STRATEGY FOR
FLUID-STRUCTURE INTERACTION

3.1 Introduction

As mentioned in Chapter 2, a transfer strategy includes two components. First, a mapping strategy is required to relate the positions of the fluid pressure points on the fluid mesh (e.g., at fluid nodes or panel centers) to the discretized structural surface and the discrete structural displacements (e.g., nodal displacements) to the discretized fluid surface. Second, interpolation is required to obtain the value of the transferred data at the required points, e.g., the structural displacements at the fluid nodes.

The transfer strategy presented here introduces a 2-D parametric space, which represents the interface between the fluid and the structure. Then, any points on the interface in the structural mesh and the fluid mesh can be mapped to the 2-D parametric space. To transfer displacements/pressures to the required points in the corresponding meshes, the SEA method (see section 2.3) is employed as the interpolation strategy. It is used to obtain a continuous representation of the displacements and pressures in the parametric space. Once a continuous field for pressure (displacement) in the parametric space is obtained, the pressure (displacement) can be transferred to the structure (fluid) mesh.

This chapter is organized as follows. First, the specific mapping strategy adopted herein is discussed. Then, displacement transfer, which is a straightforward application of
SEA, is explained. The required modification for the energy constraint to transfer pressure is discussed. Three modifications are proposed. Finally, the finite element formulation that is used to solve the variational smoothing problem is described.

3.2 Concepts of the Mapping Strategy

As discussed previously, a primary difficulty of displacement and pressure transfer is that the fluid and structural meshes define two distinct surfaces. One approach to map a point on one mesh to a point on the other mesh is the normal projection scheme, in which the surface normal vector at a point on one mesh is used to project to a point on the other mesh. However, there are several difficulties with this approach. First, in general, the structure and fluid have different normals. As shown in Figure 3.1, normal vectors $N_1$ and $N_2$ are normals to structure elements, and normal vector $N_3$ is normal to a fluid element. Second, as depicted at point $A$ on the structural mesh and point $C$ on the fluid mesh (Figure 3.1), a single point can have a non unique normal. As a result, a point on one mesh may have several projected points on the other mesh. Clearly, the question of which normal should be used for the mapping arises. In addition, a point on one mesh may have no projected points on the other mesh. For example, point $B$ on the fluid mesh has no projection on the structural mesh based on the structural normals [21].
Figure 3.1 Structure and fluid normals

It is clear that mapping from a 2-D surface to a 2-D surface is relatively straightforward. One solution for a mapping strategy for a 3-D surface is to transform it into a 2-D surface. Zhang and Luo [22] introduce an incremental finite plate element method to develop 3-D meshes of triangular finite elements to flattened forms. Its development principle is that the geometry of the flattened form is closest to the geometry of the 3-D mesh. Although this method can be adopted for arbitrary 3-D surfaces, its solution accuracy depends on the fineness of the mesh.

The mapping strategy proposed herein assumes that the actual fluid-structure interface is defined (piecewise) by analytical functions, for example, by a parametric-based CAD description of the structure. Because an analytical surface can be represented by 2-D parametric coordinates, the interface can be represented in a 2-D parametric space. It is also assumed that both the structural and fluid meshes are based on this exact geometric
description. At present, this is likely to be more common for the fluid, which requires a fairly accurate description of the surface to obtain adequate accuracy. In contrast, the structural model may represent the actual surface poorly. However, as automated meshing techniques improve and computer processing power increases, it can be anticipated that geometrically-accurate structural models will be used more frequently. As a result of the above two assumptions, any point on the interface in both meshes can be mapped to the 2-D parametric space. Consequently, discrete nodal displacements in the structure can be mapped to the parametric space. Similarly, fluid pressures at discrete points on the fluid mesh can be mapped to the parametric space.

A key aspect of this mapping strategy is to find the parametric space. The concept of the parametric space is illustrated by the following examples.

3.2.1 1-D parametric space

Figure 3.2 depicts the curve defined by the function \( x = 2(1 - y^2) \), in which \(-1 \leq y \leq 1\). The curve is also represented by the parametric equations \( x(\theta) = 2\sin^2\theta \) and \( y(\theta) = \cos\theta \), in which the parameter \( \theta \in [0, \pi] \). Hence, the 1-D parametric space is defined by \( \theta \in [0, \pi] \). If a dimensional parametric coordinate is desired, the arc length \( s \), as shown in Figure 3.2, can be used, although it is less convenient.
3.2.2 2-D parametric space

For surfaces, a 2-D parametric space can be defined if the surface is defined by analytical functions. For example, a circular cylindrical surface can be represented in cylindrical coordinates, and the two-parameter space may be defined as a 2-D rectangular space with two parameters \( z \) and \( R \phi \) (\( z \) is the longitudinal coordinate, \( \phi \) is the polar angle and \( R \) is the radius). A spherical surface can be represented in spherical coordinates, and the two-parameter space may be defined as a 2-D rectangular space with \( \phi \) and \( \theta \) where \( \phi \) is the longitude and \( \theta \) is the polar angle (see Figure 3.3). To obtain dimensional parametric coordinates, \( R \phi \) and \( R \theta \), where \( R \) is the radius of the sphere, can be used.
More complicated surfaces can also be parametrized. The Wigley hull is a well-known ship hull form in naval architecture [23]. The equation for the hull geometry is given by

\[ y = \pm \frac{B}{2} \left[ 1 - \left( \frac{2x}{L} \right)^2 \right] \left[ 1 - \left( \frac{z}{D} \right)^2 \right] \]  

(3.1)

where \( B \) is the beam, \( L \) is the length and \( D \) is the draft.

A 2-D parametric space may be defined by two parameters \( \phi \) and \( \theta \) as shown in Figure 3.4. For any \( x-y \) plane (\( z = \text{constant} \)), equation (3.1) can be written as

\[ y(\theta) = C_1 \sin^2 \theta, \quad x(\theta) = \frac{L}{2} \cos \theta \]  

(3.2a)

in which \( \theta \in [0, \pi] \) and \( C_1 = \pm \frac{B}{2} \left[ 1 - \left( \frac{z}{D} \right)^2 \right] \). For any \( y-z \) plane (\( x = \text{constant} \)), equation (3.1) can be written as
\[ y(\phi) = C_2 \sin^2 \phi, \quad z(\phi) = -D \cos \phi \] (3.2b)

in which \( \phi \in \left[ 0, \frac{\pi}{2} \right] \) and \( C_2 = \pm \frac{B}{2} \left( 1 - \left( \frac{2x}{L} \right)^2 \right) \).

Figure 3.4 Two-parameter space for a Wigley hull

To obtain dimensional parametric coordinates, a 2-D rectangular space with two parameters \( S_x \) and \( S_y \) may be chosen as the parametric representation of the surface on which \( y > 0 \), where \( S_x \) and \( S_y \) are arc lengths as shown in Figure 3.4. The surface where \( y < 0 \) is obtained by reflection of the surface where \( y > 0 \). Consider a plane perpendicular to the \( x \)-axis. All points on the intersection of the plane and the hull surface, which is a parabolic curve, have the same arc length \( S_x \). In a similar manner, consider a plane perpendicular to the \( z \)-axis. All points on the intersection of the plane and the surface have the same arc length \( S_y \). The following expressions for \( S_x \) and \( S_y \) can be obtained readily:
in which prime represents the derivative. Although equations (3.3a) and (3.3b) appear complicated, they involve straightforward calculations to obtain $S_x$ and $S_y$ given $\phi$ and $\theta$.

Of course, the geometry of practical structures is generally much more complex than in the preceding examples. Nevertheless, the definition of 2-D parametric spaces is still possible over patches as long as the surface is described piecewise analytically. Examples that use patches will be presented in Chapter 4.

Once a parametric description of the surface has been defined, parametric coordinates can be found for points on the actual surface as well as points on the meshes, because the meshes are based on the same parametric definition. Hence, the mapping procedure is straightforward. Subsequently, the discrete data mapped from one mesh to the parametric space must be interpolated to obtain the data at the required points on the other mesh. The data transfer is considered next.

### 3.3 Transfer Structural Motion to Fluid Model

The basic problem is to transfer known displacements of the nodes in the structural
mesh to nodal displacements in the fluid mesh. With the mapping strategy discussed previously, one approach would be to map the required points of the fluid mesh onto the 2-D parametric space and then find the corresponding point on the structural mesh. Interpolation functions of the structural element could then be used to find the displacement. This approach presents two difficulties. The inverse mapping from the parametric space to the structural mesh is in general more difficult than the mapping from the mesh to the parametric space. Also, it is often difficult to compute the natural coordinates of these points in the structural mesh for known global coordinates, and most FEA codes do not have this capability. SEA gives an alternative approach. Displacement transfer is a straightforward application of the unmodified SEA method (referred to herein as ‘pure smoothing’). The algorithm of the transfer scheme is outlined as follows:

- Map the nodes of the structural mesh to the parametric mesh. Let \( x_q^s \) denote the position of these nodes in the parametric space, where \( q = 1, 2, \ldots N_s \), and \( N_s \) is the total number of structural nodes.

- Use pure smoothing to obtain a \( C^1 \) continuous displacement field in the corresponding 2-D parametric space. As discussed in Chapter 2, the error functional involves scalar quantities only, and so each component of the discrete displacement data at \( x_q^s \) is smoothed independently.

- Map the nodes of the fluid mesh to the parametric mesh. Let \( x_q^f \) denote the position of these nodes in the parametric space, where \( q = 1, 2, \ldots N_f \) and \( N_f \) is the total number of fluid nodes.
• Evaluate the smoothed displacement field at \( x_q \) using the interpolation functions of the smoothing element.

3.4 Transfer Fluid Pressure From Fluid Model to Structure

The transfer of pressures from the fluid mesh to the structural mesh is essentially the reverse of the transfer of displacements. However, as mentioned, it is desirable that the transferred pressures will do the same work as the calculated fluid pressures along the fluid-structure interface. Therefore, it is necessary to modify the basic smoothing error functional in equation (2.10). Three modifications based on different ‘energy constraints’ are considered herein.

3.4.1 Error functionals

3.4.1.1 Problem definition

Let \( \mathbf{u}_s = [u_1, u_2, u_3]^T \) be the displacement at a point on the wetted surface of the structural mesh, where \( u_i \) is the displacement in direction \( x_i \). Similarly, let \( \mathbf{u}_f \) denote the displacement at a point on the wetted surface of the fluid mesh. It is assumed that

\[
\mathbf{u}_s = \sum_{i=1}^{n_{\text{mode}}} \psi_i q_i \quad \text{and} \quad \mathbf{u}_f = \sum_{i=1}^{n_{\text{mode}}} \phi_i q_i \tag{3.4}
\]

in which the \( \psi_i = [\psi_i^1, \psi_i^2, \psi_i^3]^T \) are the assumed modes and the \( q_i \) are the corresponding generalized coordinates. \( n_{\text{mode}} \) represents the total number of assumed modes. The assumed modes can be analytical functions, or if the finite element method is used to discretize the structure, they can be the finite element interpolation functions. To reduce the
number of assumed modes, the ‘dry’ normal modes of vibration (i.e., of the structure in air) are often used as the assumed modes in linear hydroelasticity. The $\phi_i$ are the assumed modes $\psi_i$ transferred to the fluid model.

With the application of equation (3.4), the virtual work of the transferred pressure $p_s$ acting on the discretized interface of the structural model is given by

$$\delta W_s = \int_{\Gamma_s} \delta \mathbf{u}_s^T \mathbf{n} p_s d\Gamma = \sum_{i=1}^{n_{mode}} \delta q_i \int_{\Gamma_s} \psi_i^T \mathbf{n} p_s d\Gamma$$  \hspace{1cm} (3.5)

where $\mathbf{n}$ is the unit normal vector to the wetted surface of the structural mesh, $\Gamma_s$, directed into the structure.

With the application of equation (3.4), the virtual work of the pressure $p_f$ acting on the interface of the fluid model is given by

$$\delta W_f = \int_{\Gamma_f} \delta \mathbf{u}_f^T \mathbf{n} p_f d\Gamma = \sum_{i=1}^{n_{mode}} \delta q_i \int_{\Gamma_f} \phi_i^T \mathbf{n} p_f d\Gamma$$  \hspace{1cm} (3.6)

where $\Gamma_f$ denotes the fluid-structure interface on the fluid model.

### 3.4.1.2 Global fluid-structure exact energy constraint

This strategy exactly enforces a global equal energy constraint, which can be represented by

$$\delta W_s = \delta W_f$$  \hspace{1cm} (3.7)

Substitution of equations (3.5) and (3.6) into (3.7) results in a constraint equation for each mode $i$:  

$$\delta W_s = \delta W_f = \sum_{i=1}^{n_{mode}} \delta q_i \left( \int_{\Gamma_s} \psi_i^T \mathbf{n} p_s d\Gamma - \int_{\Gamma_f} \phi_i^T \mathbf{n} p_f d\Gamma \right)$$
To incorporate the constraints in equation (3.8) into the basic smoothing error functional of equation (2.10), the Lagrange multiplier method is used:

\[
\Phi = \frac{1}{N} \sum_{q=1}^{N} \left[ p_{q}^f - p(x_q) \right]^2 + \alpha \sum_{e=1}^{N_p} \left[ \int_{\Omega^e} \left( (p_{x} - \Theta_x)^2 + (p_{y} - \Theta_y)^2 \right) d\Omega \right] + \beta \sum_{e=1}^{N_p} \int_{\Omega^e} \left[ (\Theta_{x,x})^2 + (\Theta_{y,y})^2 + \frac{1}{2} (\Theta_{x,y} + \Theta_{y,x})^2 \right] d\Omega 
\]

\[
+ \sum_{i=1}^{n_{mode}} \lambda_i \left[ \sum_{e=1}^{N_e} \sum_{m=1}^{n_{es}} \int_{\Gamma^e} \psi_i^T n \tilde{N}_m d\Gamma p(x_m) - \bar{F}_{fi} \right]
\]

where \( N \) is the total number of pressure data points; \( N_p \) is the total number of smoothing elements; \( N_s \) is the total number of structural elements; \( n_{es} \) is the total number of nodes for a structural element (e.g., for a 3-node element, \( n_{es} \) is three); \( \Omega^e \) denotes the smoothing element; \( x_q \) is the position vector of the discrete pressure data point; \( x_m \) is the position vector of the structural nodes in the parametric mesh; \( p(x_m) \) is the smoothed pressure at the position \( x_m \); \( \lambda_i \) is the undetermined Lagrange multiplier for mode \( i \); \( \tilde{N}_m \) are interpolation functions for the pressure distribution over the structural element; and \( \bar{F}_{fi} \) denotes the generalized force corresponding to mode \( i \) from the fluid model (i.e., the right side of equation (3.8)).

The first three terms in equation (3.9) are from equation (2.10). The fourth term in equation (3.9) imposes the energy constraint, equation (3.8), on the transferred pressure field. Note that the energy constraint is enforced globally, and that the number of addi-
tional unknowns (the Lagrange multipliers) equals the number of assumed modes. The Lagrange multiplier method increases the total number of degrees of freedom, which is unlikely to be attractive.

### 3.4.1.3 Global fluid-fluid energy constraint

The strategy here is to impose the energy constraint on the fluid model only. Specifically, the virtual work of the fluid pressure acting on the discretized interface of the fluid model is forced to be equal to the virtual work of the recovered fluid pressure acting on the same model. To satisfy this constraint, equation (3.9) is modified as

\[
\Phi = \frac{1}{N} \sum_{q=1}^{N} \left[ p_{q}^{f} - p(x_{q}) \right]^{2} + \alpha \sum_{e=1}^{N_p} \left[ \int_{\Omega^e} \left[ (p_{x} - \theta_x)^{2} + (p_{y} - \theta_y)^{2} \right] d\Omega \right] \\
+ \beta \sum_{e=1}^{N_p} \Omega_e \left[ \int_{\Omega^e} \left[ (\theta_{x,x})^{2} + (\theta_{y,y})^{2} + \frac{1}{2}(\theta_{x,y} + \theta_{y,x})^{2} \right] d\Omega \right] \\
+ \sum_{i=1}^{n_{\text{mode}}} \sum_{e=1}^{N_f} \sum_{m=1}^{n_{\text{ef}}} \int_{\Gamma_f} \hat{N}_m \phi_i^T n \hat{N}_m d\Gamma p(x_m) - \hat{F}_fi
\]

in which \(N_f\) is the total number of fluid panels; \(n_{\text{ef}}\) is the total number of nodes for a fluid panel; \(p(x_m)\) is the pressure at the fluid node in the parametric mesh; and \(\hat{N}_m\) are interpolation functions for the pressure distribution over the fluid panel. Note that this formulation does not involve the structural mesh and that the Lagrange multiplier method is used.

### 3.4.1.4 Local fluid-structure approximate energy constraint

The previous two methods enforce the energy constraint globally. The third alternative enforces the energy constraint at a ‘local’ or ‘energy-patch’ level. That is, the energy
constraint is such that the virtual work of the fluid pressure acting on a patch of the fluid model is constrained to equal the virtual work of the transferred pressure acting on the corresponding patch of the structural model. The main difficulty in this approach is to match the corresponding patches from the fluid and structural models. The ‘energy-patch’ is defined as a patch of the smoothing mesh. It is required to match the patches from the fluid and structural models with the respective patch of the smoothing mesh. Therefore, equations (3.7) and (3.8) can be interpreted as equilibrium over a patch of the smoothing mesh.

The penalty method is used to enforce these constraints, and the functional becomes

\[
\Phi = \frac{1}{N} \sum_{q=1}^{N} \left[ (p_f^q - p(x_q))^2 \right] + \alpha \sum_{e=1}^{N_p} \left[ \int_{\Omega^e} \left[ (p_{x,x} - \theta_x)^2 + (p_{y,y} - \theta_y)^2 \right] d\Omega \right] 
+ \beta \sum_{e=1}^{N_p} \int_{\Omega^e} \left[ (\theta_{x,x})^2 + (\theta_{y,y})^2 + \frac{1}{2}(\theta_{x,y} + \theta_{y,x})^2 \right] d\Omega 
+ \lambda \sum_{i=1}^{n_{mode}} \sum_{e=1}^{N_{patch}} \sum_{m=1}^{n_{es}} \left[ \int_{\Gamma_i} \mathbf{\Psi}_i^T \tilde{N}_m d\Gamma p(x_m) - F_{ei} \right]^2 
\]

in which \( n_{es} \) is the number of structural nodes in the patch; \( \tilde{N}_m \) are interpolation functions for the pressure distribution over the structural element; \( x_m \) is the position vector of the structural nodes in the parametric mesh; \( \gamma_i \) is a non-dimensionalizing scaling factor for mode \( i \); and \( \lambda \) is a non-dimensional penalty parameter. The enforcement of the energy constraint is now approximate. However, because the penalty method is used instead of the Lagrange multiplier method, no additional unknowns are involved.

The scaling factor \( \gamma_i \) is introduced in equation (3.11) so that 1) penalty parameter \( \lambda \) can be non-dimensional; and 2) the ‘energy-constraint’ term can have similar magnitude.
to the other terms in the error functional. The smoothed field, \( p(x) \), is to be recovered from the discrete data \( p_q^f \) via a penalized discrete-least square (PDLS) variational formulation. Hence, mean square value \( \frac{1}{N} \sum_{q=1}^{N} (p_q^f)^2 \) and square generalized forces \( \bar{F}_{fi}^2 \) are chosen to represent the magnitude of the first term and the fourth term in the error functional, respectively. Factor \( \gamma_i \) is determined by

\[
\gamma_i = \left[ \frac{1}{N} \sum_{q=1}^{N} (p_q^f)^2 \right] / \bar{F}_{fi}^2, \quad \text{when} \quad \bar{F}_{fi}^2 > \left[ \frac{1}{N} \sum_{q=1}^{N} (p_q^f)^2 \right] \\

\gamma_i = \gamma_{\text{max}} \quad \text{when} \quad \bar{F}_{fi}^2 \leq \left[ \frac{1}{N} \sum_{q=1}^{N} (p_q^f)^2 \right] \tag{3.12a}
\]

\[
\gamma_i = \gamma_{\text{max}} \quad \text{when} \quad \bar{F}_{fi}^2 \leq \left[ \frac{1}{N} \sum_{q=1}^{N} (p_q^f)^2 \right] \tag{3.12b}
\]

in which \( \gamma_{\text{max}} \) denotes the maximum factor.

### 3.4.1.5 Specialized error functionals

The previous formulations are essentially independent of the discretization methods used for the structural and fluid problems. The structural problem will almost always be solved by the finite element method. When the Green function (constant panel) method is used to solve the fluid problem, the second and third formulation can be specialized. In the following, the fluid is modeled as an incompressible, inviscid fluid undergoing irrotational motion, and linear potential theory is used.

For the global fluid-fluid constraint, equation (3.10) becomes
where $x_m$ is replaced by $x_c$, the center of the fluid panels.

If numerical integration is used to evaluate the integral on the left side of equation (3.8), the integral can be written as

$$
\Phi = \frac{1}{N} \sum_{q=1}^{N} \left[ (p_q - p(x))^2 + \alpha \sum_{e=1}^{N_p} \int_{\Omega'} \left[ \left( p_{x_e} - \theta_{x_e} \right)^2 + \left( p_{y_e} - \theta_{y_e} \right)^2 \right] d\Omega \right] + \sum_{e=1}^{N_p} \Omega_e \int_{\Omega'} \left[ \left( \theta_{x_e}^x \right)^2 + \left( \theta_{y_e}^y \right)^2 \right] d\Omega
$$

$$
+ \beta \sum_{e=1}^{N_p} \Omega_e \int_{\Omega'} \left[ \left( \theta_{x_e}^x + \theta_{y_e}^y \right)^2 \right] d\Omega + \sum_{i=1}^{nmodes} \sum_{e=1}^{N_f} \int_{\Gamma_f} \left[ \psi_i \n \right] p(x_c) = \sum_{g=1}^{N_g} \sum_{i=1}^{nmode} \int_{\Gamma_s} \lambda_i \n \psi_i \right]
$$

where $x_m$ is replaced by $x_c$, the center of the fluid panels.

If numerical integration is used to evaluate the integral on the left side of equation (3.8), the integral can be written as

$$
\int_{\Gamma_s} \n \psi_i \n p_s d\Gamma = \sum_{g=1}^{N_g} w_g z_i \psi_i \n p_s(x_g) J_g
$$

in which $J_g$ is the Jacobian determinant; $w_g$ is the weight appropriate to the point $g$; $x_g$ is the position vector of the integration point $g$; $N_g$ is the total number of the integration points of the structural elements that fall into one patch of the smoothing mesh; and $z_i$ is a scalar to denote $\n \psi_i$.

For the local fluid-structure constraint, the application of equation (3.14) to equation (3.11) results in
3.4.2 Algorithm for pressure transfer

The algorithm for pressure transfer is similar to the one for displacement transfer:

- Map the points on the fluid mesh where the pressure is calculated onto the parametric mesh. Let $x_q^f$ denote the position of these points in the parametric space, where $q = 1, 2,...N_f$ and $N_f$ is the total number of fluid points.
- Use smoothing methods presented previously to obtain a $C^1$ continuous pressure field in the corresponding 2-D parametric space. Again, each component of the discrete pressure data at $x_q^f$ (e.g., real part and imaginary part of the hydrodynamic pressures) is smoothed independently.
- Map the nodes of the structural mesh to the parametric mesh. Let $x_q^s$ denote the position of these nodes in the parametric space, where $q = 1, 2,...N_s$, and $N_s$ is the total number of structural nodes.
- Evaluate the smoothed pressure field at $x_q^s$ using the interpolation functions of the smoothing element.

\[
\Phi = \frac{1}{N} \sum_{q = 1}^{N} \left[ (p_q^f - p(x))^2 + \alpha \sum_{e = 1}^{N_p} \int ((p_{xe} - \theta_x)^2 + (p_{ye} - \theta_y)^2) d\Omega \right] \\
+ \beta \sum_{e = 1}^{N_p} \Omega_e \int \left[ (\theta_{xe})^2 + (\theta_{ye})^2 + \frac{1}{2}(\theta_{xe} + \theta_{ye})^2 + \frac{1}{2} \left. \partial \right|_{\Omega e}^{2} \Omega_e \right] d\Omega \\
+ \lambda \sum_{i = 1}^{N_m} \gamma_i \sum_{e = 1}^{N_p} \left[ \sum_{g = 1}^{N_g} w_g \left| z_i (x_g) J_g p(x_g) - F_{fi} \right|^2 \right]^{2} 
\]
3.4.3 Finite element formulations

As in the original SEA, the finite element method is used to minimize the error functionals in all three methods to obtain the pressure fields on the parametric domain. The finite element implementation for these methods is very similar to the smoothing element reviewed in section 2.3.2, except the element has been extended for the energy constraint terms. The basic smoothing element is a 3-node, 9 degree of freedom, triangular element. As mentioned previously, quadratic interpolation is used for pressure and linear interpolation is used for the ‘rotations’.

3.4.3.1 Global fluid-structure exact energy constraint

Minimization of the error functional in equation (3.9) results in a system of linear algebraic equations:

\[
\begin{bmatrix}
K & H^T \\
H & 0
\end{bmatrix}
\begin{bmatrix}
d \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
F \\
\bar{F}_f
\end{bmatrix}
\]  
(3.16)

in which \(K\) is the smoothing stiffness matrix expressed in equation (2.13); \(d\) is the vector containing nodal degrees-of-freedom in the smoothing domain (\(p\), and its slopes with respect to \(x\)-axis and \(y\)-axis respectively, \(\theta_x\) and \(\theta_y\)); \(F\) is the consistent load vector expressed in equation (2.15); \(\bar{F}_f\) denotes the generalized force vector; and \(\lambda\) is a vector containing Lagrange multipliers, the number of which equals the number of assumed modes. The lower partition of equation (3.16) is equation (3.8), the equation of constraint. It can be written as

\[
Hd = \bar{F}_f
\]  
(3.17)
where

\[
H = \sum_{e=1}^{N_s} (d_s^e)^T \int_{\Gamma_s^e} N_s^T \tilde{N} \alpha N(x_e) \]

(3.18)

in which \(N_s\) is the total number of structural elements; \(x_e\) is the position vector of the structural nodes of the structural element \(e\) in the parametric mesh; \(\tilde{N}\) is the interpolation function for the pressure distribution over the structural element; \(d_s^e\) is the matrix of structural nodal displacements for all modes; \(N_s\) is the structural interpolation function for normal displacement; and \(\tilde{N}(x_e)\) is the smoothing interpolation function evaluated at the \(x_e\) position.

### 3.4.3.2 Global fluid-fluid energy constraint

Minimization of the error functional in equation (3.13) results in a similar system of equations as equation (3.16), except \(H\) is given by

\[
H = \sum_{e=1}^{N_f} (d_f^e)^T A_f^e \tilde{N}(x_e) \]

(3.19)

in which \(N_f\) is the total number of fluid panels; \(d_f^e\) is the matrix of displacements at the center of fluid panel \(e\) for all modes; \(A_f^e\) is the area of the fluid panel \(e\); and \(\tilde{N}(x_e)\) is the interpolation function of the smoothing element evaluated at \(x_e\) (the center of the fluid panels).

### 3.4.3.3 Local fluid-structure approximate energy constraint

The global system of linear algebraic equations, obtained by minimizing the error
functional in equation (3.15), has the usual form

\[ \mathbf{Kd} = \mathbf{F} \]  

(3.20)

At the element level, the smoothing equations have the same basic form, \( \mathbf{K}^e \mathbf{d}^e = \mathbf{F}^e \). The element stiffness has four components, resulting from the four terms of the functional of equation (3.15):

\[ \mathbf{K}^e = \mathbf{K}_e^e + \mathbf{K}_\alpha^e + \mathbf{K}_\beta^e + \mathbf{K}_\lambda^e \]  

(3.21)

\[ = \frac{1}{N} \sum_{q=1}^{n^e} \mathbf{N}_q^T \mathbf{N}_q + \alpha \int_{\Omega^e} \mathbf{B}^T \mathbf{B} d\Omega + \beta \int_{\Omega^e} \mathbf{B}_\beta^T \mathbf{B}_\beta d\Omega + \lambda \mathbf{H}^T \mathbf{C} \mathbf{H} \]

in which \( n^e \) is the number of sample points within element \( e \); \( \mathbf{K}_\lambda^e \) is the energy-control matrix defined by the last term; and \( C_{ii} = \gamma_i, i = 1, ..., nmode, \) form the diagonal matrix \( \mathbf{C} \) of scaling factors. \( \mathbf{H} \) is defined as

\[ \mathbf{H} = \sum_{g=1}^{N_g} (\mathbf{d}_g^e)^T \mathbf{w}_g \mathbf{N}_g^T (\mathbf{x}_g) J_g \mathbf{N}(\mathbf{x}_g) \]  

(3.22)

The consistent element ‘load’ vector (right-hand side) is

\[ \mathbf{F}^e = \frac{1}{N} \sum_{q=1}^{n^e} p_q^f \mathbf{N}_q^T + \lambda \mathbf{H}^T \mathbf{C} \mathbf{F}_f \]  

(3.23)

Note that the first three terms in the element stiffness matrix have been described in section 2.3.2.
CHAPTER 4
NUMERICAL EVALUATION OF TRANSFER STRATEGIES

4.1 General Comments

This chapter is organized as follows. First, the displacement and pressure transfer strategies are assessed independently by using rigid body cases. Then, the application of the strategies to flexible bodies is presented. The pressure transfer strategies proposed in Chapter 3 have been implemented in the computer program MANOA. The implementation is based on a finite element approach to minimize the error functionals. The initial, wave-induced pressure fields are determined with the computer program HYDRAN [24], which uses a linear, constant panel, Green function formulation to obtain the wave-induced response of floating structures in the frequency domain.

For all cases, the origin of the inertial coordinate system is located on the still-water-level, with the $z$-axis directed vertically upward. Motions are defined relative to a body-fixed coordinate system that has the same orientation as the inertial-coordinate system. A wave angle of $0^\circ$ corresponds to a wave propagating in the direction of the positive $x$-axis. The wave angle is defined with the right-hand-rule as an angle about the $z$-axis. The range of wave periods from 6s to 18s is chosen for the incoming waves. The wave amplitude is denoted by $A$. Unless indicated otherwise, $A$ equals one in the following. $L$ denotes the longitudinal length. The water density $\rho$ and the displaced volume $V$ are used to nondi-
mensionalize some quantities for presentation. Infinite water depth is assumed.

4.2 Meshing Strategy

The basic smoothing element is a 3-node, 9 degree of freedom, triangular element with quadratic interpolation for primary variables and linear interpolation for the 'rotations’. However, the preferred meshing strategy is to mesh the parametric domain with quadrilateral 'macroelements’, each of which is formed by 4 triangular elements in a cross-diagonal configuration [15, 16]. Subsequently, smoothing elements refer to these macroelements. Typically, the pressure smoothing meshes use a 4-1 mapping between fluid panels and smoothing elements (see Fig. 4.1). For displacement smoothing, a 1-1 mapping between structural elements and smoothing elements is used.

![Diagram showing 1 smoothing macroelement mapped to 4 fluid panels](image_url)

Figure 4.1 A 4-1 mapping

The structural meshes are composed of shell elements, either 5-node quadrilaterals (macroelements of 4 triangles) or 3-node triangles. The fluid meshes are composed of constant pressure quadrilateral and triangular panels.
4.3 Transfer Fluid Pressure to the Structure

The proposed pressure transfer strategies are assessed by evaluating the transferred pressure fields, hydrodynamic coefficients, and motions for three rigid bodies: a box, a circular half-cylinder and a hemisphere. The different strategies are used to transfer the pressures to the structural meshes. The pressure fields, hydrodynamic coefficients, and motions determined with the structural models are then compared with those from ‘traditional’, rigid-body hydrodynamic analysis (the results from HYDRAN). It should be noted that in linear hydroelasticity, a common approach is to transfer only the displacements from the structural model to the fluid model, and then compute all quantities with the fluid model. However, the objective here is to evaluate the pressure transfer methods, and therefore the response is calculated based on the transferred pressure and the structural model. To focus on the pressure transfer, only rigid bodies are considered. This avoids the need to transfer from a flexible structural model. The six rigid body modes (surge, sway, heave, roll, pitch and yaw) are readily determined with the fluid model.

In the following examples, reference solutions are based on a fine HYDRAN mesh. Results for the fluid (HYDRAN) meshes will be denoted HYDRANX or HYX, where X is the number of panels in the model. Results obtained from the global fluid-structure method, global fluid-fluid method, and the local fluid-structure method will be denoted GF-S, GF-F, and LF-S, respectively, followed by the number of shell elements in the structural model. Results from SEA applied to stress recovery demonstrate that the smoothing results are insensitive to the penalty parameter $\alpha$ for values between 0.1 and 10000 [16]. It is also noted that it is not necessary to ensure the stability of the method when there are
sufficient sampling data [17], which is the case here. Therefore, penalty parameters $\alpha=1$ and $\beta=0$ are used.

4.3.1 Rigid box

A rigid box with plan dimensions of 90 x 90 m and a 40 m draft is a well-known benchmark [25] used in linear hydrodynamics. The center of gravity is located at (0, 0, – 10.62m). The body-fixed coordinate system is the same as the inertial-coordinate system. Incoming wave angles of 0°, 30° and 45° are considered.

4.3.1.1 Meshes

As shown in Figure 4.2a, the initial fluid model is represented by 48 quadrilateral panels. A 2-D parametric space is defined by unfolding the box. The four sides and the bottom form five patches. The parametric space is defined by the assembly of these 5 patches and is discretized by smoothing elements, as shown in Figure 4.2b.
To obtain a $C^1$ continuous recovered pressure field, clearly nodal constraints (equal pressures and tangential derivatives) must be imposed along the common edges of the patches. With reference to Figure 4.2, constraints include

\[ p_1 = p_4, \theta_{y1} = \theta_{x4} \]  
\[ p_2 = p_5 = p_8, \theta_{x2} = \theta_{x8}, \theta_{y5} = \theta_{y8}, \theta_{y2} = \theta_{x5} \]  
\[ p_9 = p_6, \theta_{y6} = \theta_{y9} \]  
\[ p_3 = p_7, \theta_{x3} = \theta_{x7} \]

where $p$ represents the nodal pressure and $\theta_x$ and $\theta_y$ are independent variables that are forced to represent the derivatives of the pressure with respect to the $x$-axis and $y$-axis, respectively. The numerical subscript is the node number. Similar constraints are required for the remaining degrees-of-freedom on the common boundaries. The implementation of nodal constraints in MANOA is detailed in Appendix I.
4.3.1.2 Results

Figure 4.3 shows the incoming pressure contour \( (N/m^2) \) for a unit amplitude wave \( (A = 1 \, m) \) with a period of 12 sec and an angle of 0°. The transferred pressure field on the structural model (GF-S96), which has 96 triangular shell elements, is in good agreement with the reference distribution, which results from a mesh that is 4 times finer than the original fluid mesh. Pressure contours from GF-F and LF-S are similar to the one from GF-S, and therefore they are not presented herein. Note that the original fluid mesh of 48 panels is quite coarse and yet the transfer methods give good results.

Clearly, the original HY48 data are very rough. To investigate the realistic performance of different methods in imposing the energy constraint, a fluid mesh with 192 panels and a structural mesh with 48 5-node quadrilateral elements (formed by 4 triangular elements) are used.

As described in Chapter 3, GF-S imposes the work done by the fluid pressure acting on the discretized interface of the fluid model to be exactly equal to the work done by the transferred pressure acting on the discretized interface of the structural model. Therefore, the hydrodynamic coefficients and exciting forces, which are calculated from the transferred pressure on the structural model, are equal to the initial HYDRAN data. As a result, the motions calculated with this method will be identical to the solutions obtained from the fluid model. Therefore, results for GF-S are not shown.

Both GF-F and LF-S are approximate methods. Because the structure and fluid meshes define identical wetted surfaces, a large penalty parameter \( (\lambda = 1000) \) could be used in LF-S. Selected added mass coefficients and generalized forces for a wave inci-
dence angle of 45° transferred by GF-F and LF-S are compared with the initial HYDRAN solution in Figure 4.4. The maximum percent difference between LF-S and HYDRAN solutions is 2.1%. These results and others show that LF-S enforces the energy constraint better than does GF-F.

Figure 4.5 shows the six motions for a wave with an angle of 30°, which are obtained from GF-F, LF-S, pure smoothing (i.e., no energy constraint), the initial HYDRAN model with 192 panels, and a reference HYDRAN model with 2112 panels. It is seen that GF-F has similar performance as pure smoothing. Neither of them captures the resonant response in sway and roll. These results demonstrate that LF-S method does a good job of transferring the pressure distribution to the structural mesh and maintaining energy conservation.

![Figure 4.5](image)

a) Real part
b) Imaginary part

Figure 4.3 Incoming wave pressure contour for a 0° wave angle

a) Surge added mass
b) Heave added mass

c) Heave exciting force (imaginary part)
d) Roll exciting moment (real part)

Figure 4.4 Added mass coefficients and exciting forces for 45° wave angle

a) Surge RAO
b) Sway RAO

c) Heave RAO
d) Roll RAO

e) Pitch RAO
f) Yaw RAO

Figure 4.5 Box motions for a 30° incoming wave angle

4.3.2 Floating half-cylinder

A floating 90m long half-cylinder with a radius of 10m is considered. The body-fixed coordinate system has an origin at the body center of gravity. The center of gravity is located at (0, 0, -4.244m). Incoming wave angles of 30° and 90° are considered.

4.3.2.3 Meshes

A fluid mesh of the half-cylinder, with 336 quadrilateral panels, was used (Figure 4.6a). The smoothing element analysis is carried out in a 2-D parametric space with 84 quadrilateral macroelements (Figure 4.6b). The structural mesh, shown in Figure 4.6c, involves 160 5-node quadrilateral shell elements and 16 triangular shell elements.
a) HYDRAN mesh (336 panels)

b) Smoothing mesh

c) Structural mesh

Figure 4.6 Floating half-cylinder

The smoothing mesh is composed of three patches, one each for the two ends and
one for the cylindrical body. As noted in section 3.2.2 and depicted in Figure 3.3, the cylindrical body section is represented easily in cylindrical coordinates. The 2-D parametric patch is defined by the longitudinal coordinate \( x \) in Cartesian coordinates and \( y = R\phi \), in which \( R \) is the radius of the cylinder and \( \phi \) is restricted to the interval \([−\pi/2, \pi/2] \).

For the semi-circular ends, one could use \( x \) and \( y \) in Cartesian coordinates as two parameters in the parametric space since the ends are 2-D surfaces. However, a problem arises with this approach. To accommodate all discrete data extracted from the fluid mesh, the smoothing mesh should be defined to be larger than the actual geometry of the ends. As a result, the continuity of the pressure and the gradients of the pressure along the common edges between the cylindrical body section and the ends can not be enforced. An alternative approach is that each semi-circular end can be represented by the rectangles shown in Figure 4.6b. \( AB \) and \( A*B* \) represent the common edge between the cylindrical body section and the end. Therefore, constraints must be imposed between \( AB \) and \( A*B* \) to ensure the continuity of pressure and derivative of the pressure with respect to the \( y \)-axis.

Let \( s \) and \( t \) be two independent variables in the parametric space. Figure 4.7 is used to illustrate the relation between the semi-circle and the rectangle. Two parameters \((s, t)\) vary on the intervals \([−\pi/2, \pi/2]R \) and \([0, R] \), respectively. Note that the parameter \( t \) is constant on each of concentric semi-circles, i.e., \( \widehat{AB} \) and \( \widehat{CD} \), while the other parameter, \( s \), varies monotonically over the intervals \([−\pi/2, \pi/2]R \). In fact, the \( t \)-axis is the polar axis in polar coordinates, which emanates from the origin \( O \). The straight line \( AB \) and \( CD \) on the rectangle correspond to semi-circles \( \widehat{AB} \) and \( \widehat{CD} \), respectively, in the physical space.
To maintain the same continuity of the pressure distribution on the rectangle as those over the semi-circular ends, some nodal constraints must be imposed. The left side $O_1O_2$ of the rectangle is mapped back to the origin $O$ in the real space. Therefore, the nodal constraints along the line $O_1O_2$ in the parametric space must be specified such that

$$p_{O_1} = p_O = p_{O_2} \quad \quad (4.2)$$

In Figure 4.7a, $r_{OA}$ lies at an angle of $\phi = -\pi/2$, while $r_{OB}$ lies in the opposite direction with the angle of $\phi = \pi/2$. Therefore, the slope at the point $O$ in the direction $OA$ should be equal to the slope at the point $O$ in the direction $OB$, but with opposite sign. The corresponding constraint equation in the parametric space is

$$\theta_{tO_1} = -\theta_{tO_2} \quad \quad (4.3)$$

where $\theta_t$ represents the derivative of the pressure with respect to the $t$-axis.

It is easy to verify this parametrization. The transformation between the two coordi-
nate systems, x-y and s-t, can be written as

\[ x = r \sin \phi = t \sin \left( \frac{s}{R} \right), \quad y = r \cos \phi = t \cos \left( \frac{s}{R} \right) \]  

(4.4)

The transformation of the derivatives of the pressure are

\[ \theta_s = \frac{r \cos \phi}{R} \theta_x - \frac{r \sin \phi}{R} \theta_y, \quad \theta_t = \sin \phi \theta_x + \cos \phi \theta_y \]  

(4.5)

where \( \theta_x \) and \( \theta_y \) represent the derivatives of the pressure with respect to the x-axis and y-axis, respectively; and \( \theta_s \) and \( \theta_t \) represent the derivatives of the pressure with respect to the s-axis and t-axis, respectively. If \( r = 0 \), from equation (4.4) both \( x \) and \( y \) are 0, which confirms that the left side \( O_1 O_2 \) of the rectangle is mapped to the origin \( O \). Substitution of \( r = 0 \) into equation (4.5) results in \( \theta_s = 0 \), which means there is uniform pressure distribution along the line \( O_1 O_2 \). It is clear that this is consistent with the true situation at the origin in the real space.

4.3.2.4 Results

Figure 4.8 shows four incoming wave pressure fields \((N/m^2)\) for a wave of unit amplitude, 10 sec period, and a 90\(^\circ\) incidence angle. The initial and reference pressure fields are denoted HY336 and HY2688, respectively. The transferred pressures, GF-S176 and LF-S176, are based on a structural mesh with 176 shell elements. As before, the transferred pressure field from LF-S (LF-S176) compares well, even with the pressure distribution from the reference solution (HY2688). It is clear that the transferred pressure field from GF-S (GF-S176) is poor. GF-F results (not shown) are similar to LF-S.
Inspection of the results reveals that the roll moment constraint causes the poor pressure distribution in GF-S176. Theoretically speaking, the roll moment for the cylindrical surface should be zero because the pressure forces and the gravity force all go through the center of the cylinder at the water surface. However, the result for the roll moment usually involves error in the numerical analysis, when the cylindrical surface is represented by a mesh with triangular elements. To illustrate this problem, a ring taken from the cylinder is shown in Figure 4.9. Normal vectors from the elements are not necessarily directed at the center of the cylinder, i.e., the normals $N_e$ of some triangular elements are shown in Figure 4.9d. The misalignment results in an error in the roll moment. Note that there are
different errors in the roll moment with different meshes. Hence, the structural mesh (Figure 4.9c) and the fluid mesh (Figure 4.9a) have different roll moments. Because GF-S enforces a global energy constraint, the transferred pressure distribution must be adjusted over the entire domain to satisfy the energy constraint in the roll mode. As a result, the solution shown in Figure 4.9c is different from what is expected in Figure 4.9b. This hypothesis was tested with identical fluid and structural meshes, for which GF-S gave a correct pressure distribution. It can be concluded that the finite element geometric discretization error causes the failure of the GF-S method to recover a good pressure distribution.

Figure 4.9 Discrepancy in normals

In Figure 4.10, selected hydrodynamic coefficients and generalized forces for a wave with an angle of incidence of 90° transferred by GF-F and LF-S are compared with the
original HYDRAN solution. The results from LF-S differ at most by 2.1% from the initial HYDRAN results. It is shown that the results from LF-S and GF-F compare well with the original HYDRAN results.

Figure 4.11 compares the six motions for a wave with an angle of 30° determined with five methods, i.e., GF-F, LF-S, pure smoothing, initial HYDRAN model and reference HYDRAN model. The five solutions are in good agreement.

Compared to the box case, the structure and fluid meshes for the cylinder define different wetted surfaces. It was found that when this happens, smaller values of the penalty parameter must be used ($\lambda = 0.1$) in LF-S.

![Graph showing surge added mass vs. wave period]

a) Surge added mass
b) Heave added mass

c) Pitch damping
d) Heave exciting force (imaginary part)

Figure 4.10 Hydrodynamic coefficients and exciting forces for 90° wave angle
a) Surge RAO

b) Sway RAO
c) Heave RAO

d) Roll RAO
Figure 4.11 Response of the half-cylinder for 30° wave angle

e) Pitch RAO

g) Yaw RAO
4.3.3 *Floating Hemisphere*

Consider a floating hemisphere with a 45m radius. The body-fixed coordinate system has an origin at the body center of gravity. The center of gravity is located at (0, 0, –16.875m). Because the smooth hemisphere is axisymmetric, the response is independent of the wave angle (note that this is not quite true for the discretized faceted body). Therefore, only the wave incidence angle of 0° is considered.

4.3.3.5 *Meshes*

The HYDRAN fluid mesh, the smoothing mesh and the structural mesh are shown in Figure 4.12. The fluid mesh has 408 quadrilateral panels and 24 triangular panels, and the 2-D smoothing mesh has 108 quadrilateral macroelements. The structural mesh has 84 5-node quadrilateral elements and 12 triangular elements. The rectangular parametric space is defined by $R\psi$ and $R\phi$, where $\psi$ is the longitude, $\phi$ is the latitude, and $R$ is the radius. The longitude $\psi$ is restricted to the interval $[0, 2\pi]$, while the latitude $\phi$ varies in the interval $[0, \pi/2]$.

Note that the points of a latitude $\phi$ form a circle of radius $R\sin\phi$, while the longitude $\psi$ varies monotonically over the interval $[0, 2\pi]$. All nodes on the top side $CA$ of the rectangular mesh are mapped to the bottom pole of the real hemisphere. All nodal pressures along that side are constrained to be equal. The left side $CD$ and the right side $AB$ represent one physical curve, and therefore constraints must be imposed along these two sides such that pressure and derivative with respect to $\phi$ are the same for the matching nodes.
a) HYDRAN mesh (432 panels)  b) Smoothing mesh

c) Structural mesh

Figure 4.12 Floating hemisphere

4.3.3.6 Results

Figure 4.13 shows the incoming pressure contours \((N/m^2)\) for a wave with a unit amplitude and a 10 sec period. In Figure 4.13a, there are four contours, which are computed from an initial fluid mesh (HY432), the reference fluid mesh (HY3600), and the structural mesh transferred by the smoothing method (GF-S and LF-S), respectively. In Figure 4.13b, the initial fluid mesh (HY432) is changed to a mesh with 216 panels.
(HY216). In that case, the transferred pressure contour is obtained from the same structural mesh in Figure 4.13a. As can be seen in Figure 4.13, the two results for GF-S method illustrate different performance; one gives a good smooth contour and the other one is poor. Similar to the cylinder, it is observed that the constraint of the pitch moment causes the problem. It is conjectured that the discretization error related to the pitch moment in the fluid and structural meshes happen to be the same in Figure 4.13a. In Figure 4.13b, the error involved with the initial fluid mesh is changed as a result of a different discretization adopted for the initial fluid model. Therefore, good performance in Figure 4.13a is just a coincidence, and the performance in both cases verifies the conclusion that the finite element discretization error causes the failure of the GF-S method to recover a good pressure distribution, which was drawn in the previous section. Again, LF-S does a good job in transferring pressure, especially when compared to a finer reference solution. GF-F gives similar pressure contour with that obtained from LF-S; therefore, the pressure contour for this method is not shown.
In Figure 4.14, selected added mass and damping coefficients and generalized forces transferred by GF-F and LF-S are compared with the initial HYDRAN solution. Although there is some discrepancy between the results obtained from the pressure on the fluid mesh and the transferred pressure on the structural mesh, the percent difference is not large. The
results from LF-S differ at most by 3.3% from the HYDRAN solutions. Note that there is a large geometrical difference between the initial fluid model (HY432) and the structural model (LF-S96). LF-S still has pretty good conservation property and has better performance than GF-F.

Figure 4.15 shows three motions (surge, heave, and pitch) determined from five methods (GF-F, LF-S, pure smoothing, initial HYDRAN model and reference HYDRAN model). As can be seen, LF-S has good agreement with the initial fluid solution.

Again, for the hemisphere, small values of the penalty parameter must be used ($\lambda = 0.1$) in LF-S.

![Graph showing surge added mass](image)

a) Surge added mass
b) Heave added mass

c) Pitch damping
d) Surge exciting force (real part)

e) Heave exciting force (imaginary part)
f) Pitch exciting moment (real part)

Figure 4.14 Hydrodynamic coefficients and exciting forces

a) Surge RAO
Figure 4.15 Response of the hemisphere for $0^\circ$ wave angle
4.3.4 Discussion

Based on the numerical results presented, the following conclusions may be made.

1) General application of GF-S is limited to configurations with flat surfaces if an accurate pressure distribution is required. 2) GF-F is the least effective in imposing the energy constraint. 3) LF-S is the most robust formulation. Even for relatively large geometric differences between fluid and structural meshes, LF-S can perform well.

LF-S appears to be the most promising strategy. In addition to GF-S’s sensitivity to geometric differences in the meshes, it also introduces additional unknowns in the smoothing analysis through the Lagrange multipliers. This can be problematic for problems with a large number of degrees of freedom. LF-S introduces no additional unknowns. It has been shown that the penalty parameter $\lambda$ has to be fairly small if the geometry described by the structural mesh is different from the geometry described by the fluid mesh.

4.4 Transfer Structural Motion to Fluid Model

In hydroelasticity, it is necessary to transfer the displacements from the structural models to the fluid model. If the assumed modes include rigid body modes, it usually is not necessary to transfer them because they can be computed readily based on the fluid mesh. However, the objective here is to evaluate the displacement transfer, and therefore the response is calculated based on the transferred rigid body modes on the fluid model. Only rigid bodies are considered currently to allow a direct comparison with the results from the ‘traditional’ approach, wherein the rigid modes are defined directly on the fluid
The displacement transfer strategy is evaluated by using the previous three rigid body models (a box, a circular half-cylinder and a hemisphere). The structural meshes and fluid meshes used previously are employed here as well. For the smoothing mesh, a 1-1 mapping between structural elements and smoothing elements is used.

In the following examples, results for the fluid (HYDRAN) meshes will be labeled HYX, where X is the number of panels in the model. Results based on displacement modes, which are transferred from the structural model to the fluid model, will be labeled SMTHX-Y, where X and Y are the number of structural elements and fluid panels, respectively. Penalty parameters $\alpha = 1$ and $\beta = 10^{-7}$ or $10^{-8}$ are used.

### 4.4.1 Rigid box

The mapping for the box is exact and straightforward. Both the fluid and structural meshes represent the wetted surface exactly. Because the rigid body modes are linear functions of both physical coordinates and parametric coordinates, smoothing gives the exact rigid body modes, and all quantities computed with the transferred modes are identical to the reference quantities.

### 4.4.2 Floating half-cylinder

Figure 4.16 shows the six motions for the half-cylinder for a wave with an incidence angle of $30^\circ$. It can be seen that the results compare very well.
a) Surge RAO

b) Sway RAO
c) Heave RAO

d) Roll RAO
e) Pitch RAO

f) Yaw RAO

Figure 4.16 Cylinder RAOs for 30° wave angle
4.4.3  Floating hemisphere

Surge, heave and pitch RAOs for the hemisphere are shown in Figure 4.17. Again, the results based on the transferred displacement modes agree very well with the reference solutions.
Figure 4.17 Hemisphere RAOs for 0° wave angle
CHAPTER 5

APPLICATION TO FLEXIBLE BODIES

5.1 General Comments

As mentioned before, nonlinear, time-domain hydroelastic analysis of flexible off-
shore structures requires that the structural motion be transferred to the fluid model and
the resulting fluid pressure at the fluid-structure interface be transferred from the fluid
model to the structure. Application of the methodology to two flexible bodies, a barge and
a cylinder, is presented. The flexible bodies are analyzed as follows. First, the assumed
modes including rigid body modes are determined with the structural models. Second,
these modes are transferred to the fluid meshes. The wave-induced pressure fields are then
determined by HYDRAN based on those transferred modes on the fluid model. These
pressure fields are transferred to the structural meshes. Finally, the response is calculated
based on the transferred pressures and the structural models. The total pressure fields,
motions and stresses determined with the structural models are then compared with those
from HYDRAN models.

The displacement field on or in the structure is represented by \( u = [u_1, u_2, u_3]^T \),
where \( u_i \) is the displacement in direction \( x_i \). It is assumed that

\[
\mathbf{u} = \sum_{i=1}^{n} \mathbf{\psi}_i p_i \tag{5.1}
\]

The \( \mathbf{\psi}_i = [\psi_{1,i}, \psi_{2,i}, \psi_{3,i}]^T \) are assumed modes and the \( p_i \) are the corresponding normal
coordinates. In the following examples, normal coordinates are presented to compare the effect of displacement and pressure transfer between the structural models and HYDRAN models.

In the following examples, results for the fluid (HYDRAN) meshes will be labeled HY-\(\theta\)-mode\(X\) or HY-\(\theta\), where \(\theta\) is angle for the incoming wave and \(X\) is the number of the assumed mode. Those results are based on transferred displacements modes, which are transferred from the structural model to the fluid model. Results for the structural model will be labeled M-\(\theta\)-mode\(X\) or M-\(\theta\), which are based on transferred pressure from the fluid model. Penalty parameters \(\alpha = 100\) and \(\beta = 10^{-7}\) or \(10^{-8}\) are used.

### 5.2 A flexible barge

A 30.48 m (100 ft) long flexible, box shaped barge with a 1.22 m (4 ft) draft is studied. The cross section is 7.3 m (24 ft) wide by 1.83 m (6 ft) high. The center of gravity is located at \((0, 0, 0.15 \text{ m})\). The geometric and material properties are

\[
E = 7.9 \times 10^4 \text{ kN/m}^2, \quad \nu = 0.3, \quad t = 0.03 \text{ m}, \quad \rho_1 = 9.42 \text{ kN/m}^3, \quad \rho_2 = 3.07 \text{ kN/m}^3
\]

\[
(E = 1.65 \times 10^6 \text{ psf}, \quad \nu = 0.3, \quad t = 0.1 \text{ ft}, \quad \rho_1 = 59.95 \text{ pcf}, \quad \rho_2 = 19.5344 \text{ pcf})
\]

where \(t\) is the thickness of the plate; \(\rho_1\) is the mass density of the top plate and \(\rho_2\) is the mass density of the bottom plate. It is assumed that the four sides are massless and the total mass of the structure is distributed in the top plate and the bottom plate. The Young’s modulus \(E = 7.9 \times 10^4 \text{ kN/m}^2\) has been used so that the flexible modes of the barge can be obtained within the range of wave period chosen for the incoming wave. There are continuous longitudinal bulkheads at intervals of 1.22 m (4 ft) and transverse bulkheads at inter-
vals of 1.524 m (5 ft). These bulkheads are assumed to be massless for the dry modal analysis, but of finite thickness, so as to model the stiffness property of the structure. Wave angles of 0°, 30°, 45° and 90° are considered. The range of wave periods chosen for the incoming waves is 2s to 18s.

5.2.1 Meshes

A fluid mesh of the barge, with 1096 quadrilateral panels, was used (Figure 5.1a). The structural mesh, shown in Figure 5.1b, involves 396 5-node quadrilateral shell elements. Displacement transfer is carried out in a 2-D parametric space with 224 quadrilateral macroelements (Figure 5.1c). The smoothing mesh for pressure transfer, shown in Figure 5.1d, involves 336 quadrilateral macroelements.
b) Structural mesh

c) Smoothing mesh for displacement transfer
d) Smoothing mesh for pressure transfer

Figure 5.1 Flexible barge

5.2.2 Displacement transfer

Ten assumed modes are determined with the structural model. The first six modes correspond to the traditional rigid body modes. The four flexible modes are vertical bending (Figure 5.2a), twist (Figure 5.2b), 2nd vertical bending (Figure 5.2c) and horizontal bending (Figure 5.2d), respectively. First three rigid body modes are scaled by taking a unit displacement in three translational directions respectively. Three rotational rigid body modes are scaled by taking a unit degree per meter for three rotations respectively. Four flexible modes are scaled by taking a unit displacement at the bow (–15.24 m, 0, –1.22 m). The transferred flexible modes on the fluid model (shown in Figures 5.3a, b, c and d) agree very well with the original modes.
a) Mode7-vertical bending

b) Mode8-twist

c) Mode9-2nd vertical bending
d) Mode10-horizontal bending

Figure 5.2 Flexible modes with the structural model

a) Mode7-vertical bending
b) Mode8-twist

c) Mode9-2nd vertical bending
d) Mode10-horizontal bending

Figure 5.3 Flexible modes with fluid model

5.2.3 Pressure transfer

Figure 5.4a and Figure 5.4b show the real part and the imaginary part of the total pressure field for a 5 sec wave at an incidence angle of 30°. The initial and transferred pressure fields are denoted HY1096 and STRUCT396, respectively, where the number denotes the number of elements in each model. The transferred pressure fields are in good agreement with the original pressure fields from the HYDRAN model.
Figure 5.4 Total pressure contour for a 5sec wave at $30^\circ$.

Figure 5.5a to Figure 5.5j compare normal coordinates per unit incoming wave.
amplitude determined for two models and for the different normal modes. Figure 5.6a and Figure 5.6b compare the vertical displacement per unit incoming wave amplitude at the bow (–15.24 m, 0, –1.22 m) and the mid-length (0, 0, –1.22 m) respectively with two models. Figure 5.7a and Figure 5.7b show the normal stress $\sigma_{11}$ at two points (0, 0, –1.22 m) and (7.62 m, 0, –1.22 m) determined with two models, where $\sigma_{11}$ is along the longitudinal direction of the barge. It can be seen that there is very good agreement between these two sets of results.

![Graph of wave period vs amplitude with various line styles and markers for different modes.](image)

a) Surge (m/m)
b) Sway (m/m)

c) Heave (m/m)
d) Roll (degree/m)

![Roll Graph]

e) Pitch (degree/m)

![Pitch Graph]
f) Yaw (degree/m)

g) Vertical bending
h) Twist

i) 2nd vertical bending
j) Horizontal bending

Figure 5.5 Normal coordinates

a) Vertical displacement at (0,0,–1.22 m)

94
b) Vertical displacement at \((-15.24 \text{ m}, 0, -1.22 \text{ m})\)

Figure 5.6 Comparison of vertical displacement

a) Normal stress at \((0,0,-1.22 \text{ m})\)
b) Normal stress at (7.62 m, 0, –1.22 m)

Figure 5.7 Comparison of normal stress $\sigma_{11}$

5.3 A flexible half-cylinder

A 30.48 m (100 ft) long flexible half-cylinder with a radius of 3.66 m (12 ft) and a 1.95 m (6.4 ft) draft is studied. The center of gravity is located at (0, 0, 0.15 m). The material properties are

\[ E = 7.9 \times 10^4 \text{ kN/m}^2, \quad \nu = 0.3, \quad t = 0.03 \text{ m}, \quad \rho = 7.8 \text{ kN/m}^3 \]

\[ (E = 1.65 \times 10^6 \text{ psf}, \quad \nu = 0.3, \quad t = 0.1 \text{ ft}, \quad \rho = 49.6 \text{ pcf}) \]

where $t$ is the thickness of the hull; $\rho$ is the mass density of the cylindrical surface. It is assumed that the total mass of the structure is distributed in the top plate and all other plates are massless. The Young’s modulus $E = 7.9 \times 10^4$ kN/m$^2$ has been used so that the flexible modes of the cylinder can be obtained within the range of wave period chosen for
the incoming wave. There are continuous longitudinal bulkheads at intervals of 1.22 m (4 ft) and transverse bulkheads at intervals of 1.52 m (5 ft). These bulkheads are assumed to be massless for the dry modal analysis, but of finite thickness, so as to model the stiffness property of the structure. Wave incidence angles of 0° and 90° are considered. The range of wave periods chosen for the incoming waves is from 3 s to 18 s.

5.3.1 Meshes

The fluid mesh has 440 quadrilateral panels and 16 triangular panels (Figure 5.8a). The structural mesh has 308 five-node quadrilateral elements and 32 triangular elements (Figure 5.8b). Displacement transfer is carried out in a 2-D parametric space with 208 quadrilateral macroelements (Figure 5.8c). The smoothing mesh for pressure transfer, shown in Figure 5.8d, involves 118 quadrilateral macroelements.

a) Fluid Mesh
b) Structural Mesh

c) Smoothing mesh for displacement transfer

d) Smoothing mesh for pressure transfer

Figure 5.8 Flexible half-cylinder
5.3.2 Displacement transfer

Ten assumed modes are determined with the structural model. First six modes correspond to the traditional rigid body modes. Four flexible modes are vertical bending (Figure 5.9a), horizontal bending (Figure 5.9b), twist (Figure 5.9c), and 2nd vertical bending (Figure 5.9d). First three rigid body modes are scaled by taking a unit displacement in three translational directions respectively. Three rotational rigid body modes are scaled by taking a unit degree per meter for three rotations respectively. Four flexible modes are scaled by taking a unit displacement at the bow (−15.24 m, 0, −1.95 m). Figure 5.10a to Figure 5.10d show the four transferred flexible modes on the fluid model. It can be seen that the results compare very well.

a) Mode 7-vertical bending
b) Mode 8-horizontal bending

c) Mode 9-twist
d) Mode10-2nd vertical bending

Figure 5.9 Flexible modes with the structural model

a) Mode7-vertical bending

b) Mode8-horizontal bending
5.3.3 Pressure transfer

Figure 5.11a and Figure 5.11b show the real part and the imaginary part of the total pressure field for a 5 sec wave at an angle of 0°. The initial and transferred pressure fields are denoted HY456 and STRUCT340, respectively, where the number denotes the number of elements in each model. The transferred pressure fields are in good agreement with the original pressure fields on the HYDRAN model.
a) Real part

b) Imaginary part

Figure 5.11 Total pressure contour for a 5sec wave at 0°

From Figure 5.12a to Figure 5.12h, normal coordinates per unit wave amplitude for
the different normal modes determined with the structural model are compared with the HYDRAN results. Figure 5.13a and Figure 5.13b compare the vertical displacement at the bow (−15.24 m, 0, −1.95 m) and the mid-length (0, 0, −1.95 m) respectively with two models. Figure 5.14a and Figure 5.14b show the normal stress $\sigma_{11}$ at two points (0, 0, −1.95 m) and (7.62 m, 0, −1.95 m) determined with two models, where $\sigma_{11}$ is along the longitudinal direction of the hull. It can be seen that there is very good agreement between these two sets of results.
b) Sway (m/m)

c) Heave (m/m)
d) Roll (degree/m)

e) Pitch (degree/m)
f) Vertical bending

![Vertical bending graph]

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![Wave period graph]

g) Horizontal bending

![Horizontal bending graph]
h) 2nd vertical bending

Figure 5.12 Normal coordinates

a) Vertical displacement at (0,0,–1.95 m)
b) Vertical displacement at \((-15.24 \text{ m}, 0, -1.95 \text{ m})\)

Figure 5.13 Comparison of vertical displacement

a) Normal stress at \((0,0,-1.95 \text{ m})\)
b) Normal stress at (7.62 m, 0, –1.95 m)

Figure 5.14 Comparison of normal stress $\sigma_{11}$
CHAPTER 6

THE HYDROSTATIC STIFFNESS OF FLEXIBLE FLOATING STRUCTURES

6.1 Problem Definition

The problem considered here involves the changes in hydrostatic pressure forces and the structural forces as a result of quasi-static displacements of the structure. With no loss in generality, the global coordinate system \((x_1, x_2, x_3)\) is defined such that the origin is on the still-water-plane and \(x_3\) is directed upward. The following assumptions are made.

1. The floating structure is at rest in a calm fluid.
2. The structure is in equilibrium. Specifically, the external forces (applied loads and the resisting hydrostatic pressure) and the internal forces in the structure constitute a statically consistent system of forces. Small (infinitesimal) displacements about the equilibrium configuration are considered. The static equilibrium configuration is assumed to be the initial (i.e., ‘undeformed’) configuration, which is consistent with linear hydroelasticity.
3. The fluid is incompressible and has a constant mass density \(\rho\). Hence, the hydrostatic pressure for \(x_3 \leq 0\) is \(p = -\rho \ g \ x_3\), where \(g\) is the gravitational acceleration.
4. The magnitude and direction of the applied loads are independent of the displacements.
5. The applied loads result from gravity forces only. They are represented at point \(x\) by the body force vector \(\mathbf{b} = [0, 0, b_3]^T = [0, 0, -\rho_s \ g]^T\), where \(\rho_s\) is the mass density of the structure. This assumption is made for convenience only; other loads can be
readily accommodated.

6  The displacement field on or in the structure is represented by \( \mathbf{u} = [u_1, u_2, u_3]^T \), where 
\( u_i \) is the displacement in direction \( x_i \). It is assumed that

\[
\mathbf{u} = \sum_{i=1}^{n} \psi_i q_i
\]  
(6.1)

The \( \psi_i = [\psi_1^i, \psi_2^i, \psi_3^i]^T \) are assumed-modes and the \( q_i \) are the corresponding generalized coordinates. The assumed-modes can be analytical functions, or if the finite element method is used to discretize the structure, they can be the finite element interpolation functions. Often in linear hydroelasticity, a subset of the normal modes of vibration of the structure ‘in-air’ are used for the \( \psi_i \).

As stated previously, the external forces result from the hydrostatic pressure on the wetted surface and the structural weight. In the static equilibrium configuration, the distributed hydrostatic pressure force at point \( \mathbf{x} \) on the wetted surface \( S_o \) is given by 

\[ \mathbf{p} = -\rho \mathbf{g} x_3 \mathbf{N}, \]

where \( \mathbf{N} \) is a unit vector normal to the structural surface and directed into the fluid. The generalized external force, \( F_i^E \), corresponding to mode \( \psi_i \) is

\[
F_i^E = \int_{S_o} \psi_i \cdot \mathbf{p} \, dS + \int_{\Omega_i} \psi_i \cdot \mathbf{b} \, d\Omega
\]  
(6.2)

where \( \Omega_i \) is the structural volume in the equilibrium configuration. The generalized internal force, \( F_i^I \), corresponding to mode \( \psi_i \) is

\[
F_i^I = \int_{\Omega_i} \sigma_{kl}^i \varepsilon_{kl}^i \, d\Omega
\]  
(6.3)

in which \( \sigma_{kl}^i \) are the actual stresses and \( \varepsilon_{kl}^i \) are the strains compatible with mode \( \psi_i \). (The
summation convention is used throughout for repeated indices.) Equilibrium requires
\[ F^E_i = F^I_i \] (6.4)

6.2 Rigid Body Modes

The traditional rigid body modes of surge, sway, heave, roll, pitch, and yaw are defined herein as the first six modes. Specifically, for point \( x \),

\[
\Psi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Psi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
\Psi_4 = \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}, \quad \Psi_5 = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad \Psi_6 = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}
\]

(6.5a)

where roll, pitch, and yaw are defined relative to the origin of the coordinate system. Higher modes (\( \Psi_i, i \geq 7 \)) are deformational modes.

6.3 Previous Formulations

As stated previously, at least three explicit formulations for the hydrostatic stiffness coefficients can be found in the literature. A commonly used form is the one proposed by Price and Wu [10]:

\[
K_{fi} = -\rho g \int_{S_i} \psi_i^j \psi_k^i N_k dS
\]

(6.6)

Newman [26] proposed a different expression:
These formulations consider only the hydrostatic pressure terms. Therefore, they cannot give the complete hydrostatic stiffness coefficients for rigid body motion, because these coefficients depend on the structural weight as well. Newman [26] discussed this relative to equation (6.7). For rigid body motion, equation (6.7) gives the correct stiffness coefficients related to the hydrostatic pressure. This latter point can be verified by substitution of equations (6.5a) and (6.5b). To obtain a form which results in the correct rigid body stiffness, Riggs [27] added a weight term:

\[
K_{fi} = -\rho g \int \frac{x_3 \psi^j_{l,l} \psi^l_k}{S_o} dS - \rho g \int \frac{\psi^j_k \psi^l_k}{S_o} N_k dS \tag{6.7}
\]

although it was recognized at the time as still being incomplete. That equation (6.8) provides the complete hydrostatic stiffness for rigid body motion can be verified by substitution of equations (6.5a) and (6.5b) into (6.8). All three formulations lead to an unsymmetric stiffness in general.

### 6.4 Complete Hydrostatic Stiffness

The complete hydrostatic stiffness is derived herein by a consistent linearization of the external and internal generalized forces. The derivation extends an earlier one [27], and it corrects an error therein.

The hydrostatic stiffness \(K_f\) represents the first order (linear) variation in the general-
ized forces as a result of small displacements from the equilibrium configuration. It can be decomposed into two components:

\[ \mathbf{K}_f = \mathbf{K}^f + \mathbf{K}^g \]  

(6.9)

in which \( \mathbf{K}^f \) results from the external forces and \( \mathbf{K}^g \) results from the internal stresses. This latter component is the well-known geometric stiffness matrix (or initial-stress stiffness matrix).

### 6.4.1 External generalized force

The stiffness coefficients \( K_{ij}^f \) represents the change in \( F_i^E \) as a result of a small displacement in the pattern \( \psi_j \). Let \( \mathbf{x}' = \mathbf{x} + \varepsilon \psi_j \), where \( \varepsilon \) is a small scalar parameter. Then

\[ F_i^E = \rho g \int_{S} \mathbf{x}_j' \cdot \mathbf{n} \, ds + \int_{\Omega_j} \psi_i \cdot \mathbf{b} \, d\Omega \]  

(6.10)

in which \( \mathbf{n} \) is the unit normal vector to the actual wetted surface \( S \). The integral over the deformed area \( ds \) can be transformed to the undeformed configuration by the relation [28]

\[ \mathbf{n} ds = J(F^{-1})^T \mathbf{N} dS \]  

(6.11a)

\( \mathbf{F} \) is the deformation gradient and \( J \) is the determinant of \( \mathbf{F} \). Considering only the first order term of equation (6.11a) results in

\[ n_k ds = [\delta_{kl}(1 + \varepsilon \psi_{m,m}^j) - \varepsilon \psi_{l,k}^j]N_l dS \]  

(6.11b)

in which \( \delta_{kl} \) is the Kronecker delta. Substitution of equation (6.11b) into (6.10) results in

\[ F_i^E = \rho g \int_{S} (x_3 + \varepsilon \psi_3^j) \psi_k^j [\delta_{kl}(1 + \varepsilon \psi_{m,m}^j) - \varepsilon \psi_{l,k}^j]N_l dS + \int_{\Omega_j} \psi_k^j b_k d\Omega \]  

(6.12)
The stiffness coefficients can be obtained from a consistent linearization of $F_i^E$ via the directional derivative [29]. That is,

$$K_{ij}^f = -\nabla F_i^E \cdot \psi_j \equiv -\frac{d}{d\varepsilon}([F_i^E(x + \varepsilon \psi_j)])_{\varepsilon = 0} \quad (6.13)$$

where the negative sign reflects the fact that the stiffness and the external forces are on opposite sides of the equations of motion.

Application of equation (6.13) to (6.12) results in

$$K_{ij}^f = -\rho g \int_{S_o} \psi_i^i \psi_j^j N_k dS - \rho g \int_{S_o} x_3 \psi_i^i \psi_j^j N_k dS + \rho g \int_{S_o} x_3 \psi_i^i \psi_j^j N_k dS \quad (6.14)$$

As an aside, if the pressure is constant, rather than varying with depth, then the first term in equation (6.14) disappears and $-\rho g x_3$ in the remaining two terms is replaced by the constant pressure. The resulting expression is equivalent to the one derived in a different manner by Hibbitt [30].

### 6.4.2 Internal generalized force

The geometric stiffness matrix, $K^g$, represents the change in the forces required for equilibrium when existing internal stresses are subject to small displacements. The formulation of the geometric stiffness is well-known. However, it will be derived here with the same procedure as that used to derive $K^f$.

Consider the $i$-th generalized internal force as a result of a displacement in the $j$-th mode. To express the volume integral in the initial, static equilibrium configuration, it is convenient to use the second Piola-Kirchhoff stress tensor, $\tilde{T}$, and the Green-Lagrange strain. It is assumed that $\tilde{T}$ does not vary as a result of the displacement. Furthermore,
because small displacements are considered, only the first order term of the Green-Lagrange strain is used:

\[
F^l_i = \int_{\Omega_s} \tilde{T}_{lm} (\delta_{km} + \epsilon \psi^j_{k,m}) \psi^l_{k,j} d\Omega
\]  

(6.15)

Because the initial configuration is assumed to be the undeformed configuration, the second Piola-Kirchhoff stress is equal to the initial Cauchy stress, \(\sigma\). Hence, \(\sigma\) can be used in equation (6.15). Once again, the directional derivative can be used to obtain the geometric stiffness coefficients:

\[
K^g_{ij} = \int_{\Omega_s} \sigma_{lm} \psi^i_{k,l} \psi^j_{k,m} d\Omega
\]  

(6.16)

From the symmetry of the stress tensor, it is clear from equation (6.16) that \(K^g\) is symmetric. Equation (6.16) is equivalent to the general formulation for geometric stiffness derived in a different manner by Cook et al. [31].

### 6.4.3 Complete hydrostatic stiffness

Substitution of equations (6.14) and (6.16) into (6.9) results in

\[
K_{fij} = -\rho g \int_{S_o} \psi^l_i \psi^j_l N_k dS + \rho g \int_{S_o} x_3 \psi^l_i \psi^j_l N_k dS + \int_{\Omega_s} \sigma_{lm} \psi^l_i \psi^j_l d\Omega
\]

(6.17)

in which \(\epsilon^l_v \equiv \psi^l_{l,l}\) is the volumetric strain. Equation (6.17) is an explicit expression for the hydrostatic stiffness coefficients. It is applicable to both rigid body motion and flexible (deformable) motion, and to the entire structure as well as at the finite element level.
6.5 Characteristics of $K_f$

In this section, several characteristics of the general formulation will be shown. First, the symmetry of $K_f$ for a floating body will be demonstrated. Then, it will be shown that equation (6.17) represents the correct hydrostatic stiffness for rigid body motion. Finally, equation (6.17) will be specialized for beam elements, which results in the beam geometric stiffness based on effective tension.

6.5.1 Symmetry of hydrostatic stiffness

To show that $K_f$ is symmetric, it is necessary only to show that $K^f$ is symmetric, because the symmetry of the geometric stiffness matrix was noted above. Stokes’ theorem can be used to integrate the last two terms of equation (6.14) to shift the differentiation from $\psi_j$ to $\psi_f$. For a closed surface $S_o$, the following general relation can be established from Stokes’ theorem:

$$\int_{S_o} g_{f_{k,\text{i}} N_k dS} - \int_{S_o} g_{k,f_{l,\text{i}} N_k dS} = -\int_{S_o} g_{l,f_{k,\text{i}} N_k dS} + \int_{S_o} g_{k,f_{l,\text{i}} N_k dS}$$

(6.18)

in which $f(x)$ and $g(x)$ are continuous and differentiable on the surface. If the actual wetted surface $S_o$ is closed by including the free surface enclosed by $S_o$, the second two terms in equation (6.14) are unchanged because $x_3 = 0$. Application of equation (6.18) to these terms gives

$$\int_{S_o} (x_3 \psi_l^i) \psi_{k,\text{i}} N_k dS - \int_{S_o} (x_3 \psi_k^i) \psi_{l,\text{i}} N_k dS$$

(6.19)

$$= -\int_{S_o} \psi_3 \psi_k^i N_k dS - \int_{S_o} x_3 \psi_l^i \psi_k^i N_k dS + \int_{S_o} \psi_k^i \psi_3 N_k dS + \int_{S_o} x_3 \psi_k^i \psi_l^i N_k dS$$
Substitution of equation (6.19) into (6.14) results in

\[ K_{ij}^f = -\rho g \int_{S_0} \psi_j^i \psi_k^i N_k dS - \rho g \int_{S_0} x_3 \psi_j^i \psi_k^i N_k dS + \rho g \int_{S_0} x_3 \psi_j^i \psi_k^i N_k dS \]  

(6.20)

Comparison of equations (6.14) and (6.20) shows the symmetry of \( K_f \) for the entire structure. However, it should be noted that the contribution to \( K_f \) from an individual finite element can be unsymmetric if the pressure is not continuous along the element’s surface, which is almost always the case. This will be demonstrated later.

### 6.5.2 Specialization for rigid body motion

It is important to demonstrate that equation (6.17) results in the correct hydrostatic stiffness coefficients for rigid body motion. For rigid bodies, the internal stresses are in general indeterminate, and equation (6.17) cannot be applied directly. We show therefore that equation (6.17) is equivalent to the well-known form for rigid bodies. The volume integral in equation (6.17) (i.e., equation (6.16)) can be integrated by parts to obtain

\[ K_{ij}^g = \int_{S_0} \sigma_{ml} \psi_j^i \psi_k^i N_m dS - \int_{\Omega_0} \sigma_{ml,m} \psi_j^i \psi_k^i d\Omega \]  

(6.21)

in which the symmetry of the stress tensor has been exploited. The surface tractions are related directly to the hydrostatic pressure:

\[ \sigma_{ml,m} N_m = \rho g x_3 N_l \]  

(6.22)

Also, the equations of local equilibrium are well-known:

\[ \sigma_{ml,m} + b_l = 0 \]  

(6.23)

Finally, the zero strain condition for rigid body motion is
Substitution of equations (6.22), (6.23) and (6.24) into (6.21) results in

\[ K_{ij}^g = -\rho g \int_{S_0} x_3 \psi_{l,k}^i \psi_{k}^j N_l dS + g \int_{\Omega_s} \rho_s \psi_{3,k}^i \psi_{k}^j d\Omega \]  

(6.25)

in which \( b_3 = -\rho_s g \). Replacement of the volume integral in equation (6.17) with (6.25) results in equation (6.8), which as stated gives the correct hydrostatic stiffness coefficients for a rigid body.

### 6.5.3 Specialization for beam elements

Beam elements are frequently used to model structures, and it is therefore interesting to specialize equation (6.17) to beams and compare the resulting formulation to the formulation which is typically used. Beam stiffnesses are formulated in a local coordinate system, e.g., \( x_1 - x_2 - x_3 \), in which \( x_1 \) corresponds to the longitudinal axis of the beam and the other two axes are orthogonal, transverse axes. The derivation in this section is based on this local coordinate system. In the following, the finite element interpolation functions involve: 1) linear interpolation for the axial displacements; 2) standard cubic Hermitian polynomials for the transverse displacements; and 3) zero shear deformation.

Before proceeding, it is necessary to discuss the geometric stiffness matrix for beam elements. In typical applications, only the axial stress, \( \sigma_{11} \), is considered. When this assumption is used in equation (6.16), the ‘usual’ beam geometric stiffness matrix, which is derived in a different manner, does not result. There are additional nonzero terms involving the axial displacements. These terms are discussed by Cook et al. [31] in the
context of bar elements. In addition, the bending terms are of the form $K_{ij}[1 + (B/L)^2]$, where $B$ is the depth, $L$ is the length, and $K_{ij}$ are the usual coefficients. For a beam, $(B/L)^2 \ll 1$, and therefore these terms can usually be ignored. All of these additional terms result from $\sigma_{11} \psi_1^j \psi_1^i$ in the integral, which in turn results from the use of the Green-Lagrange strain in the derivation. If this term is neglected and the integration over the cross section is carried out, the usual beam geometric stiffness matrix results, i.e.,

$$K_{ij}^g = \int_L P_a \psi_{k,1}^j \psi_{k,1}^i d\bar{x}_1, \ k \neq 1$$  (6.26)

in which $P_a$ is the actual tension. In offshore engineering, the actual tension is replaced by the effective tension [32] for submerged elements, where for our case the effective tension is $P_{eff} = P_a + pA_e$, $p$ is the constant pressure, and $A_e$ is the exterior cross sectional area.

In the following derivation, it is assumed that 1) the beam cross section is constant, 2) the sides are exposed to hydrostatic pressure, and 3) the ends are either ‘wet’ or ‘dry’, depending on whether or not they are exposed to the hydrostatic pressure. An element would normally be connected to adjacent elements at dry ends. In local coordinates, the hydrostatic pressure is $p = p_o - \rho g T_{3i} \bar{x}_l$, in which $p_o$ is the pressure at the origin of the local coordinate system and $T_{ij}$ are the direction cosines which transform a vector from local coordinates to global coordinates. Hence, $p_{ij} = -\rho g T_{3i}$. If the transverse shear stresses ($\sigma_{12}, \sigma_{13}$) and the terms $\sigma_{kk} \psi_{k,k}^i \psi_{k,k}^j$ are ignored, equation (6.16) becomes

$$K_{ij}^g = \int_{\Omega_i} \sigma_{11} \psi_{k,1}^j \psi_{k,1}^i d\Omega + \int_{\Omega_i} \sigma_{lk} \psi_{k,1}^i \psi_{l,1}^j d\Omega, \ k, l \neq 1$$  (6.27)

The second term in equation (6.27) can be integrated by parts to obtain
\[ K_{ij}^g = \int_{\Omega_s} \sigma_{11} \psi_{k,1}^i \psi_{k,1}^j d\Omega + \int_s (p \psi_{1,2}^i \psi_{1,2}^j N_2 + p \psi_{1,3}^i \psi_{1,3}^j N_3) dS, \quad k \neq 1 \]  

(6.28)

in which the relationship between the surface tractions and the hydrostatic pressure has been exploited, and \( s \) is the surface of the element.

Turning to the hydrostatic pressure term, equation (6.14) in local coordinates is

\[ K_{ij}^f = \int_{S_0} p \psi_{k,1}^i \psi_{k,1}^j N_k dS + \int_s p \psi_{k,1}^i \psi_{k,1}^j N_k dS - \int_s p \psi_{k,1}^i \psi_{k,1}^j N_k dS 
- \int_{S_{de}} p \psi_{k,1}^i \psi_{k,1}^j N_k dS + \int_{S_{de}} p \psi_{k,1}^i \psi_{k,1}^j N_k dS \]  

(6.29)

in which the surface \( S = S_0 \cup S_{de} \). \( S_{de} \) is the surface of the dry ends. Substitution of equations (6.28) and (6.29) into (6.9) and application of the divergence theorem to the terms which are related to the surface \( S \) result in

\[ K_{fij} = \int_L P_{eff} \psi_{k,1}^i \psi_{k,1}^j d\bar{x} + \int_{S_0} p \psi_{1,1}^i \psi_{1,1}^j N_1 dS + \int_{S_{de}} p \psi_{1,1}^i \psi_{1,1}^j N_1 dS, \quad k \neq 1 \]  

(6.30)

where the coupling terms between axial displacements and bending displacements are omitted because they result from the change of element’s length.

The first term is simply the usual geometric stiffness with the effective tension based on the external pressure evaluated at the center of the cross section. (For pipes with an internal pressure, the same procedure as above can be applied.) In practice, equation (6.30) will be preferable to the more general formulation because it does not require knowledge of the transverse normal stresses.
6.5.4 Specialization for plate elements

The procedure of specialization for plate elements is similar to the derivation for beam elements. Plate stiffnesses are formulated in a local coordinate system, e.g., \( \bar{x}_1 - \bar{x}_2 - \bar{x}_3 \), in which \( \bar{x}_3 \) is a direction normal to the plane and the other two axes are orthogonal coordinates in the plane. The derivation in this section is based on this local coordinate system. In the following, points on the midsurface \( \bar{x}_3 = 0 \) move in only the \( \bar{x}_3 \) direction as the plate deforms in bending, and transverse shear deformation is assumed to be zero. Thus the finite element interpolation functions are based on Kirchhoff plate theory.

In the following derivation, it is assumed that 1) only one of \( \bar{x}_1 - \bar{x}_2 \) planes, top or bottom, is exposed to hydrostatic pressure, 2) the other plane and four sides are dry, and 3) Only normal stress, which results from hydrostatic pressure, and membrane stresses \( (\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}) \) are considered in the geometric stiffness matrix for plate elements. An element would be connected to adjacent elements at dry sides. Equation (6.16) can be divided into two parts: the first part involves only membrane stresses, which is the ‘usual’ plate geometric stiffness matrix; the second one is related to terms of normal stress \( \sigma_{33} \) associated with respective strains, and it can be integrated by parts so that equation (6.16) becomes

\[
K_{ij}^g = K_{ij}^{usual} - \int_S (f(p)\psi_1^i\psi_1^j,_{3}N_3 + f(p)\psi_2^i\psi_2^j,_{3}N_3)\,dS
\]  

(6.31)

in which the relationship between the surface tractions and the hydrostatic pressure has been exploited, \( S \) is the surface of the element, and \( f(p) \) is a continuous function that
varies linearly with $\bar{x}_3$ from one plane exposed to hydrostatic pressure to the other dry plane through the volume of the element.

For the plate element, just as what we do for the beam element, equation (6.14) in local coordinate can be written as

$$K_{ij}^f = \int_{S_0} f(p) \psi_3^i \psi_3^j N_3 dS + \int_{S} f(p) (\psi^{i}_{l,l} - \psi^{j}_{l,l}) N_k dS$$

$$- \int_{S_{de}} f(p) (\psi^{i}_{l,l} - \psi^{j}_{l,l}) N_k dS$$

in which $S_{de}$ is the surface of four sides and $S_0$ is the wet surface. Substitution of equations (6.31) and (6.32) into (6.9) and application of the divergence theorem to the terms which are related to the surface $S$ result in

$$K_{ij} = K_{eff}^g + \int_{S_0} p_j \psi_3^i N_3 dS - \int_{S_{de}} f(p) \psi_3^j N_k dS , k = 1, 2$$

$$K_{eff}^g =$$

$$\int [(p/2 + N_1) \psi_3^i \psi_3^j + (p/2 + N_2) \psi_3^i \psi_3^j + N_{12} \psi_3^i \psi_3^j + N_{21} \psi_3^i \psi_3^j] dA$$

where $p$ is the hydrostatic pressure as defined in the previous section, and $N_1, N_2, N_{12}, N_{21}$ are membrane forces corresponding to membrane stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}$ respectively, which are defined by $N_{ij} = \int_{-t/2}^{t/2} \sigma_{ij} d\bar{x}_3$.

From equations (6.30) and (6.33), we can tell that unsymmetry results from the last term. This term cancels with the corresponding term from an adjacent element, at least when the elements are connected rigidly. This point is very significant in practical applications.
6.6 Examples

In this section the hydrostatic stiffness coefficients for some simple structural configurations will be evaluated. These examples are discussed to illustrate the formulation and to demonstrate several significant issues.

6.6.1 Vertical column modeled by one element

Consider the floating vertical column in Fig. 6.1 with the dimensions shown. The column just pierces the still water line. The column is rectangular, with a thickness $D$ (out-of-plane dimension). Only two-dimensional motion, in the $x_1$-$x_3$ plane, will be considered. Assume that the column is weightless, but that a weight $W = \rho g B D L$ acts on the top of the column. The column is modeled by a single beam finite element, which has six degrees-of-freedom: axial displacement, transverse displacement, and a rotation at the top and bottom nodes. The displacements $u_1$ and $u_3$ within the element are obtained from the nodal displacements $d$ through the matrix of interpolation functions (expressed in global coordinates):

![Figure 6.1 Floating Column](image-url)
The interpolation functions, which are used as the assumed modes, are based on the same assumptions stated in the previous section. Evaluation of equation (6.14) results in the 6 x 6 symmetric matrix $K_f$:

$$
\begin{align*}
\begin{bmatrix}
0 & 1 - 2\frac{x_3^3}{L^3} - 3\frac{x_3^2}{L^2} & \frac{x_3^3}{L^2} + 2\frac{x_3^2}{L} + x_3 & 0 & 2\frac{x_3^3}{L^3} + 3\frac{x_3^2}{L^2} & \frac{x_3^3}{L^2} + \frac{x_3}{L} \\
1 + \frac{x_3}{L} - x_1\left(-6\frac{x_3^2}{L^2} - 6\frac{x_3}{L^2}\right) - x_1\left(3\frac{x_3^2}{L^2} + 4\frac{x_3}{L} + 1\right) - \frac{x_3}{L} - x_1\left(6\frac{x_3^2}{L^2} + 6\frac{x_3}{L}\right) - x_1\left(3\frac{x_3^2}{L^2} + 2\frac{x_3}{L}\right)
\end{bmatrix}
d
\end{align*}
$$

The geometric stiffness matrix is obtained from equation (6.16). In this case the non-zero stresses are the axial stress, $\sigma_{33}$, and the transverse stress, $\sigma_{11}$. The column is assumed to be ‘solid,’ and therefore $\sigma_{33} = -\rho g L$. The transverse stress is assumed to be...
independent of \( x_1 \) and equal to the surface stress, i.e., \( \sigma_{11} = \rho g x_3 \). The resulting 6 x 6 geometric stiffness matrix is

\[
K^g = \rho g BD
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{9}{5} & \frac{L}{5} & 0 & \frac{9}{5} & \frac{L}{10} \\
0 & \frac{L}{5} & \frac{L^2}{6} & 0 & \frac{-L}{5} & \frac{L^2}{20} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{9}{5} & \frac{L}{5} & 0 & \frac{9}{5} & \frac{L}{10} \\
0 & \frac{L}{10} & \frac{L^2}{20} & 0 & \frac{-L}{10} & \frac{7L^2}{30}
\end{bmatrix}
\]

(6.37)

in which the terms related to \( \sigma_{33} \psi^i_{3,3} \psi^j_{3,3} \) have been ignored.

The symmetric hydrostatic stiffness matrix, \( K^f \), is the sum of equations (6.36) and (6.37). The stiffness coefficients for rigid body motion can be obtained from \( K^f \) by applying corresponding nodal displacements. The nonzero coefficients which result are

\[
K_{f \text{heave}} = \mathbf{d}_h^T K_f \mathbf{d}_h = \rho g BD
\]

(6.38)

\[
K_{f \text{pitch}} = \mathbf{d}_p^T K_f \mathbf{d}_p = -\frac{1}{2} \rho g BD L^2 + \frac{1}{12} \rho g B^3 D
\]

(6.39)

in which \( \mathbf{d}_h \) and \( \mathbf{d}_p \) are the nodal displacements in heave and pitch, respectively. Clearly, equations (6.38) and (6.39) are the correct rigid body stiffness coefficients.

The hydrostatic stiffness matrix can also be obtained from equation (6.30), with

\[
P_{\text{eff}} = -\rho g BD(L + x_3) \cdot \text{The terms } \rho g BD \text{ and } \rho g B^3 D/12 \text{ in } K_{f44} \text{ and } K_{f66}, \text{ respec-}
\]
tively, result from the surface integral at the bottom of the column.

### 6.6.2 Vertical column modeled by two elements

We now model the previous column with two beam elements, both of length \( L/2 \). For convenience, we only consider the transverse displacement and rotation at the three nodes, so there are 6 degrees-of-freedom. The hydrostatic pressure component for the top and bottom elements, \( k_1^f \) and \( k_2^f \), respectively, are obtained from equation (6.14):

\[
k_1^f = \rho g BD \begin{bmatrix}
6 & \frac{L}{5} & \frac{L^2}{60} & \frac{L^2}{120} \\
\frac{L}{10} & \frac{L}{10} & \frac{L}{10} & \frac{L^2}{120} \\
\frac{6}{5} & \frac{6}{5} & \frac{L}{2} & \frac{L^2}{20} \\
0 & \frac{L^2}{120} & 0 & \frac{L^2}{20}
\end{bmatrix}
\]  

(6.40)

\[
k_2^f = \rho g BD \begin{bmatrix}
\frac{18}{5} & \frac{7L}{10} & \frac{18}{5} & \frac{L}{10} \\
\frac{L}{5} & \frac{L^2}{12} & \frac{L}{5} & \frac{L^2}{40} \\
\frac{18}{5} & \frac{L}{5} & \frac{18}{5} & \frac{L}{10} \\
\frac{L}{10} & \frac{L^2}{40} & \frac{L}{10} & \frac{7L^2}{60} + \frac{B^2}{12}
\end{bmatrix}
\]  

(6.41)

Note the unsymmetry of these matrices, which is a result of the discontinuity of the pressure along the elements’ surfaces. The structural stiffness matrix \( K^f \) can be assembled from \( k_1^f \) and \( k_2^f \). This matrix is symmetric, because
This example demonstrates the cancellation of the unsymmetry by adjacent elements.

### 6.6.3 Thin shell-bending

The pressure-related integrals in equation (6.17) involve displacements and their derivatives that are to be evaluated on the wetted surface. When the structure is modeled by shell finite elements, it may be more convenient to evaluate these components along the midplane. It is therefore interesting to investigate the order of errors which may result from using midplane values. Only the hydrostatic pressure component is affected. Consider a square ‘element’ on the bottom hull of a structure. The element has a length $\alpha L$, where $0 < \alpha \leq 1$ and $L$ is a characteristic length of the structure. The element is defined by the points $(x_1, x_2, x_3)$ such that $x_1 \in [-\alpha L, \alpha L]$, $x_2 \in [-\alpha L, \alpha L]$, and $x_3 \in [-d, -d+t]$, and the wetted surface is at $x_3 = -d$. For demonstration purposes, assume the midplane transverse displacement in the element is defined by the bubble function

$$
\psi_3^f = \frac{1}{(\alpha L)^4}(x_1 + \alpha L)(x_1 - \alpha L)(x_2 + \alpha L)(x_2 - \alpha L)
$$

which is zero along the boundary and a maximum at the center. The surface displacements are obtained from equation (6.43) based on zero shear deformation. The exact nonzero stiffness coefficients $K_{ij}^f$ are

$$
K_{34}^f = k_1^f(3, 4) + k_2^f(1, 2) = k_1^f(4, 3) + k_2^f(2, 1) = K_{43}^f
$$

(6.42)

$$
K_{7i}^f = \frac{16}{9} \rho g (\alpha L)^2 
$$

(6.44a)

$$
K_{77}^f = \frac{256}{225} \rho g (\alpha L)^2 \left[ 1 + 5 \left( \frac{d}{\alpha L} \left( \frac{t}{\alpha L} \right) \right) \right]
$$

(6.44b)
The coefficients based on the midplane displacements and strains are the same, except the term involving \( t \) in equation (6.44b) is missing. That is, the error is the term involving \( t \) and \( d \). If \( \alpha \) is equal to 1, equation (6.43) represents ‘global’ bending. The error is likely to be acceptable because the term involving \( t/L \) and \( d/L \) in equation (6.44b) is likely small. If \( \alpha \ll 1 \), equation (6.43) represents local bending. Although \( t/(\alpha L) \) is still likely small, the last term in equation (6.44b) may be significant because \( d/(\alpha L) \) may be large.

### 6.6.4 Shallow draft floating plate

Recently, attention has been devoted to using shallow draft theory and thin plate theory to model mat-like floating airports; see, e.g., Kim and Ertekin [33]. It is therefore interesting to explore the hydrostatic stiffness which results from equation (6.17) for this type of model. Let the plate have thickness \( t \) and draft \( d \). Let \( L \) be a characteristic length. For example, it might be the actual length, or for a very large plate it might be one-half the length of propagating waves. For illustration, a very simplified, idealized model will be used. Consider a plate with dimensions \( L \times L \times t \), and assume that the weight of the plate is uniformly distributed and acts on the top surface. The total weight is equal to the total buoyancy force, \( \rho g d L^2 \). On the plate perimeter, the membrane stresses \( \sigma_{11} \) and \( \sigma_{22} \) are equal to the hydrostatic pressure. They vary linearly on the submerged surface and are zero on the surface above the still water plane. However, for simplicity, assume that the membrane stresses are constant through the thickness, and that the total force is equal to the total hydrostatic pressure force. Specifically, assume that

\[
\sigma_{11} = \sigma_{22} = -\frac{1}{2} \rho g \frac{d^2}{t} \quad (6.45a)
\]
Consider rigid body heave, rigid body pitch, and the deformation described by equation (6.43). The nonzero hydrostatic stiffness coefficients are

\[ \sigma_{33} = -\rho gd \]  
\[ K_{33} = \rho g L^2 \]  
\[ K_{55} = \frac{1}{12} \rho g L^4 \left[ 1 + 6 \left( \frac{d}{L} \right)^2 - 12 \left( \frac{d}{L} \right) \left( \frac{t}{L} \right) \right] \]  
\[ K_{37} = K_{73} = \frac{121}{144} \rho g L^2 \left[ 1 + \frac{24}{11} \left( \frac{d}{L} \right) \left( \frac{t}{L} \right) - \frac{24}{11} \left( \frac{d}{L} \right)^2 \right] \]  
\[ K_{77} = \frac{41209}{57600} \rho g L^2 \left[ 1 - \frac{162400}{41209} \left( \frac{d}{L} \right)^2 + \frac{146160}{41209} \left( \frac{d}{L} \right) \left( \frac{t}{L} \right) \right. \]  
\[ \left. + \frac{31200}{41209} \left( \frac{d}{L} \right)^4 - \frac{62400}{41209} \left( \frac{d}{L} \right)^3 \left( \frac{t}{L} \right) + \frac{90080}{123627} \left( \frac{d}{L} \right)^2 \left( \frac{t}{L} \right)^2 \right] \]

It can be readily verified that the stiffness coefficients for rigid body heave and pitch are correct.

Many analyses appear to make the assumption that the draft is zero; i.e., the initial stresses are zero. When this is done, the terms involving \( d \) and \( t \) in the above equations disappear. Although rigid body heave is unaffected, the stiffness coefficients for rigid body pitch and bending are no longer correct. Whether the missing terms are significant or not will depend on the actual application.

### 6.7 Comparison with Previous Formulations

In this section the hydrostatic stiffness coefficients determined by Price and Wu’s formulation and Newman’s formulation for two examples are shown. These examples are
used to illustrate the comparison between the complete formulation presented here and the previous formulations in the literature.

### 6.7.1 Vertical column modeled by one element

The floating vertical column in section 6.6.1 is considered. The problem definition is same as described before. The interpolation functions are defined by equation (6.35). Price and Wu’s formulation, equation (6.6), results in the 6x6 skew-symmetric matrix $K_f$:

$$K_f = \rho g BD$$

The stiffness coefficients for rigid body motion can be obtained from $K_f$ by applying corresponding nodal displacements. The nonzero coefficients which result are

$$K_{f_{heave}} = d_h^T K_f d_h = \rho g BD$$

$$K_{f_{pitch}} = d_p^T K_f d_p = -\frac{1}{2} \rho g BDL^2 + \frac{1}{12} \rho g B^3 D$$

in which $d_h$ and $d_p$ are the nodal displacements in heave and pitch, respectively.

Note that the matrix in equation (6.47) is different from the sum of equations (6.36)
and (6.37). It doesn’t include the effect of the structural weight, and the authors give no indication of how to include the weight term. However, equations (6.48) and (6.49) give the correct rigid body stiffness coefficients because it is assumed that the weight acts on the top of the column, which is the origin of the coordinates in this case. As a result, the contribution of the weight to the rigid body stiffness coefficients, referenced to the top of the column, is zero.

Newman’s formulation, equation (6.7), results in the 6x6 unsymmetric matrix $\mathbf{K}_f$:

$$
\mathbf{K}_f = \rho g BD
$$

\[
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & \frac{3}{5} & -\frac{L}{10} & 0 & \frac{3}{5} & -\frac{B^2}{2L} \\
0 & \frac{L}{10} & \frac{L^2}{30} & 0 & \frac{L}{10} & -\frac{L^2}{60} + \frac{B^2}{6} \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & \frac{3}{5} & \frac{L}{10} & 0 & \frac{3}{5} & \frac{B^2}{2L} \\
0 & 0 & \frac{L^2}{60} & 0 & L & \frac{L^2}{10} + \frac{5B^2}{12}
\end{bmatrix}
\] (6.50)

Again, the stiffness coefficients for rigid body motion can be obtained from $\mathbf{K}_f$ by applying corresponding nodal displacements. The nonzero coefficients which result are

$$
K_{f\text{heave}} = \mathbf{d}_h^T \mathbf{K}_f \mathbf{d}_h = \rho g BD
$$

$$
K_{fpitch} = \mathbf{d}_p^T \mathbf{K}_f \mathbf{d}_p = -\frac{1}{2} \rho g BD L^2 + \frac{1}{12} \rho g B^3 D
$$

Clearly, the matrix in equation (6.50) is different from the sum of equations (6.36)
and (6.37). It doesn’t include the effect of the structural weight, and the author gives no indication of how to include the weight term. Newman [26] discussed this relative to equation (6.7). However, equations (6.51) and (6.52) give the correct rigid body stiffness coefficients because it is assumed that the weight acts on the top of the column, which is the origin of the coordinates in this case.

### 6.7.2 Shallow draft floating plate

The shallow draft floating plate in section 6.6.4 is used to compare the hydrostatic stiffness coefficients from equations (6.6) and (6.7) with the results obtained with the current formulation.

The nonzero hydrostatic stiffness coefficients from Price and Wu’s formulation, equation (6.6), are

\[
K_{33} = \rho g L^2
\]  
(6.53a)

\[
K_{55} = \frac{1}{12} \rho g L^4 \left[ 1 - 6 \frac{d}{L} \right]^2
\]  
(6.53b)

\[
K_{37} = \frac{121}{144} \rho g L^2
\]  
(6.53c)

\[
K_{73} = \frac{121}{144} \rho g L^2 \left[ 1 + \frac{24}{11} \frac{(d/L)(t/L)}{L} - \frac{24}{11} \left( \frac{d}{L} \right)^2 \right]
\]  
(6.53d)

\[
K_{77} = \frac{41209}{57600} \rho g L^2 \left[ 1 - \frac{73080}{41209} \left( \frac{d}{L} \right)^2 + \frac{73080}{41209} \frac{(d/L)(t/L)}{L} \right]
\]  
(6.53e)

The nonzero hydrostatic stiffness coefficients from Newman’s formulation, equation (6.7), are
It is interesting to note that \( K_{73} \) in equation (6.53d) from Price and Wu’s formulation and \( K_{37} \) in equation (6.54c) from Newman’s formulation agree with the result in equation (6.46c) from the current formulation. However, in general, it is shown that neither of these two formulations gives the correct hydrostatic stiffness matrix. If it is assumed that the draft is zero using shallow draft theory, all the terms involving \( d \) in the above equations disappear. Then, the results from two previous formulations agree with the result from the complete formulation presented here. If \( t/L \ll 1 \), the terms involving \( d \) and \( t \) in the above equations may be insignificant. As a result, if only considering the transverse displacement, all formulations give similar results.

### 6.8 Implementation of Hydrostatic Stiffness in MANOA

The hydrostatic stiffness formulation has been implemented for two finite elements in MANOA. These two finite elements are MIN3S and MIN5S. MIN3S is a Mindlin trian-
gular, linear shell element. MIN5S is a Mindlin quadrilateral linear shell element consisting of four MIN3S (triangular) elements in a cross-diagonal pattern.

The hydrostatic stiffness has two components as stated in equation (6.9). According to the discussions in Section 5.5.1, it should be noted that $\mathbf{K}^f$ at an individual finite element level will be unsymmetric, although the total structure $\mathbf{K}^f$ for a floating structure will be symmetric because of cancellations. For convenience, the element matrices $\mathbf{K}^f$ are symmetrized by averaging the corresponding off-diagonal terms, which results in the correct structure $\mathbf{K}^f$ after assembly. For a MIN3S or MIN5S shell element, only the membrane stresses are considered in equation (6.16) and these existing internal stresses are subject to both in-plane displacements and transverse displacements. Specifically, the subscripts $l$ or $m$ in equation (6.16) are equal to 1 or 2, and $k$ is equal to 1, 2, or 3, where 1, 2 and 3 denote the local coordinate system $(x, y, z)$ respectively.
CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Transfer strategy

A general methodology to transfer displacements and pressures in loosely-coupled fluid structure interaction analyses, such as occur in hydroelasticity and aeroelasticity, has been detailed. When the structural model and the fluid model describe the same wetted surface, the data transfer is fairly straightforward. However, in practice, the structural model is often different from the fluid model. In such cases, the interfacing strategy becomes important. The strategy consists of two parts: mapping points between disparate meshes and data transfer (interpolation). The mapping strategy proposed herein is based on both meshes having the same underlying parametric description. Within the limits of this assumption, the strategy appears to be applicable to a wide class of structural shapes. The interpolation strategy is based on smoothing element analysis. For pressure transfer SEA has been modified to include an equal-energy constraint and three different formulations (GF-S, GF-F, and LF-S) of the constraint have been examined. LF-S appears to be the most promising strategy.

The transfer method has been tested by examining the performance for three floating rigid bodies: a box, a cylinder, and a hemisphere. To evaluate the displacement transfer, the displacement modes were transferred from the structural to the fluid mesh, and the transferred modes were used to compute the response. Results agreed well with the refer-
ence solutions using exact displacement modes. To evaluate the pressure transfer, the pressure from the fluid model was transferred to the structural model, and exciting forces, hydrodynamic coefficients, and motions were obtained. These results also agree well with those from the fluid model.

Application of the methodology to two flexible bodies, a barge and a floating cylinder, has been presented. The finite element mode shapes were transferred from the structure to the fluid mesh and the resulting fluid pressures on the wetted surface were transmitted from the fluid model to the structure. Responses were obtained from the structural model and agreed well with those from the fluid model.

The interfacing strategy for fluid-structure interaction appears to be very promising. It is recommended that the following work be carried out:

1. Apply the method to more complex geometrical shapes, such as one would find in practical floating structures.
2. Develop a user-friendly input system for the analysis.

7.2 Hydrostatic stiffness matrix

An explicit formulation for the hydrostatic stiffness matrix of flexible structures, for use in linear hydroelastic analysis, has been derived based on a consistent linearization of the generalized external and internal forces. It is applicable to both floating and restrained structures, although the focus herein has been on floating structures. The current formulation gives the correct hydrostatic stiffness coefficients for rigid body motion, and it results in a symmetric hydrostatic stiffness matrix of the structure.
The following additional comments can be made:

1. The hydrostatic stiffness matrix has been defined as the sum of one component that involves the hydrostatic pressure forces and one component that is the well-known geometric stiffness matrix of the structure. This definition is consistent with the one used in linear, rigid body hydrodynamics, and it contains that formulation as a special case.

2. Although the hydrostatic stiffness matrix of a floating structure is symmetric, individual element hydrostatic stiffness matrices may not be symmetric. Unsymmetry results if the hydrostatic pressure is not continuous over the entire element surface.

3. The general formulation requires the transverse normal stresses, which result from the hydrostatic pressure, for the geometric stiffness of beam elements. However these are in general not known, and one can more conveniently use effective tension in the usual beam geometric stiffness (i.e., the present formulation is consistent with the usual practice).

4. It may be necessary to include the transverse normal stress for shell elements, although it is likely that for many problems ignoring this term will result in small errors only.

5. For thin shell elements, the use of midplane values for displacements can lead to errors. The displacements on the surface should be used in evaluating the hydrostatic pressure component of the stiffness matrix.
APPENDIX I

IMPLEMENTATION OF NODAL CONSTRAINTS

Basic boundary conditions are specified in MANOA by the bcid command. This command allows degrees-of-freedom of each node to be specified to be ‘free’, restrained to be zero, or constrained to be equal to another degree-of-freedom. The bcid command creates the array .bcid(7, #nodes). A value of 0 in .bcid indicates the corresponding DOF is free, and a 1 indicates the displacement is zero. A negative value means the displacement for the degree-of-freedom of this node is the same as for the corresponding degree-of-freedom of node number |value|. More complex nodal constraints cannot be specified by the bcid command, and as a result a new command, nodal_constraints, has been implemented to expand the capabilities of MANOA.

General kinematic constraints enforce a relationship between two or more DOFs. The constrained degree-of-freedom depends on the independent degrees-of-freedom via the linear constraint equation:

\[ d_c = \alpha_1 d_1^i + \alpha_2 d_2^i + \ldots \]  

(I.1)

in which \( d_c \) denotes the constrained DOF, \( \alpha_i \) are numerical factors and \( d_i^i \) denotes the independent DOF \( i \).

The user specifies constraints in the form of equation (I.1) with the nodal_constraints command. This command modifies the array .bcid such that the value 2 is inserted in .bcid corresponding to constrained DOFs. It also creates the array .const_nodes(maxnodes,#constraints), in which the constraints are stored column-wise. The first value in each
column is the node number of the constrained DOF, while the remaining values are the node numbers associated with the independent DOFs.

The equations are numbered based on the restraint codes in .bcid. The nodal equation numbers are put in .node_eqs(6,#nodes). If the code in .bcid shows the DOF is constrained, i.e., has the value 2, the negative of the constraint number is put in the array .node_eqs. The constraint number is obtained by searching the array .const_nodes(maxnodes,#constraints).

The global stiffness or mass matrix is formed by assembling element matrices. When a mass or stiffness matrix for a constrained element is to be assembled, the element actually sends the unconstrained matrix to the global assembly routine. It also sends the mapping vector lg(ndof). For an unconstrained DOF, lg contains the global equation number corresponding to the local DOF. For a constrained DOF, the lg vector contains the negative of the constraint number. In this case, a new mapping vector, lg2(ndof2), is formed. ndof2 is the new number of DOFs of the element. For a value less than zero in lg(ndof), the independent DOFs $d_I$ associated with the constrained DOF $d_c$ are found, and the equation numbers of the corresponding independent DOFs are inserted in lg2(ndof2). At the same time, a transformation matrix tr(ndof,ndof2) is formed. For an unconstrained DOF, zeros are put in the corresponding row of the transformation matrix except for a 1 in the location corresponding to the unconstrained DOF. For the constrained DOFs, the factors $\alpha_i$ are inserted according to the corresponding independent DOFs $d_I^i$. Accordingly, element displacements and forces are transformed by

$$d = T \dd, \bar{F} = T^T F$$  \hspace{1cm} (I.2)
in which $\bar{d}$ is the displacement vector of independent DOFs; $\mathbf{d}$ is the displacement vector of the actual DOFs; and $\mathbf{F}$ and $\mathbf{F}'$ are the corresponding load vectors. As a result, the following transformation can be obtained:

$$\mathbf{k} = \mathbf{T}^T \tilde{\mathbf{k}} \mathbf{T} \quad (I.3)$$

in which $\tilde{\mathbf{k}}$ is the element stiffness matrix for a constrained element, i.e., in terms of the independent DOFs. The transformation of mass matrices is similar.

To illustrate the strategy described above, consider the simple four bar structure of Figure I.1.

With only axial deformation allowed, each bar has axial stiffness $k = EA/L$. Assume that the constraint $u_2 = u_1 + 2u_4$ is to be imposed. The structural equation of element number 3 is

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & k & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix} \quad (I.4)$$

The mapping vectors for this element are

$$\begin{bmatrix} lg \\ lg2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} lg \\ lg2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad (I.5)$$
equation (I.2) becomes

\[
\begin{bmatrix}
  u_2 \\
  u_3
\end{bmatrix} = \begin{bmatrix}
  1 & 2 & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_4 \\
  u_3
\end{bmatrix}
\]

\[\text{(I.6)}\]

equation (I.3) becomes

\[
k_3 = \begin{bmatrix}
  1 & 2 & 0 \\
  0 & 0 & 1
\end{bmatrix}^T \begin{bmatrix}
  k & -k \\
  -k & k
\end{bmatrix} \begin{bmatrix}
  1 & 2 & 0 \\
  0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
  k & 2k & -k \\
  2k & 4k & -2k \\
  -k & -2k & k
\end{bmatrix}
\]

\[\text{(I.7)}\]

The user guide for the MANOA command nodal_constraint follows.

Command Syntax

```
nodal_constraint  #constraints=? maxnodes=?
[type=general] c=n cdof=cdof r=n1,n2,\ldots rdof=r1,r2,\ldots factor=f1,f2,\ldots
```

- `type` is the type of the constraint (default = `general`)
- `#constraints` is the total number of constraints
- `maxnodes` is the maximum number of nodes for any constraint (default=2)
- `c` is the node number with a constrained DOF
- `cdof` is the node-local, constrained degree-of-freedom (1-6)
- `r` is the node numbers with independent DOFs
- `rdof` is the degrees of freedom for nodes `r`
- `factor` is the numerical factor in the constraint equation

The constraint information is stored columnwise for each constraint in

- `.const_nodes(maxnodes,#constraints) = n,n1,n2,\ldots`
- `.const_dofs(maxnodes,#constraints) = cdof,r1,r2,\ldots`
- `.const_factor(maxnodes,#constraints) = l,f1,f2,\ldots`

The constrained degree-of-freedom depends on the independent degrees-of-freedom via the constraint equation:

\[
\text{constrained dof} = f1*n1(r1) + f2*n2(r2) + \ldots
\]

in which \(n_i(r_j)\) represents the \(r_j\)th displacement of node \(n_i\). These displacements must be independent; i.e., they cannot be constrained by another constraint equation.

This is an optional command. The command `bcid` must be processed prior to this command. This command modifies the array `.bcid(7,# nodes)`, created by the `bcid` command, such that the value 2 is inserted in `.bcid` corresponding to constrained DOFs. This command, if used, must be processed prior to `num_eqs` and `form_k`. 

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End input with a blank line.

See also
bcid form_k num_eqs pdisp
REFERENCES


